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A PRESENTATION FOR THE PURE HILDEN GROUP

Stephen Tawn

Abstract. Consider the half ball, $\mathbb{B}^3_+$, containing n unknotted and unlinked arcs $a_1, a_2, \ldots, a_n$ such that the boundary of each a_i lies in the plane. The Hilden (or Wicket) group is the mapping class group of $\mathbb{B}^3_+$ fixing the arcs $a_1 \cup a_2 \cup \cdots \cup a_n$ setwise and fixing the half sphere $\mathbb{S}^2_+$ pointwise. This group can be considered as a subgroup of the braid group on $2n$ strands. The pure Hilden group is defined to be the intersection of the Hilden group and the pure braid group. In a previous paper, we computed a presentation for the Hilden group using an action of the group on a cellular complex. This paper uses the same action and complex to calculate a finite presentation for the pure Hilden group. The framed braid group acts on the pure Hilden group by conjugation and this action is used to reduce the number of cases.

1. Introduction

Let $\mathbb{B}^3_+$ be a half ball in upper half space and let $\mathbb{S}^2_+$ be the half sphere contained in its boundary. The half ball and half sphere intersect the plane in a 2-ball $\mathbb{B}^2$ and a circle $\mathbb{S}^1$. We can embed n disjoint semi-circular arcs, $a_1, a_2, \ldots, a_n$, into $\mathbb{B}^3_+$ so that the arcs are disjoint from $\mathbb{S}^2_+$ and only intersect $\mathbb{B}^2$ at their end points (see Figure 1). We will write $a = a_1 \cup a_2 \cup \cdots \cup a_n$. The Hilden group $\mathbf{H}_{2n}$ is the group of isotopy classes of self-homeomorphisms of $\mathbb{B}^3_+$ which preserve a setwise and $\mathbb{S}^2_+$ pointwise. The inclusion $(\mathbb{B}^2, \partial a, \mathbb{S}^1) \hookrightarrow (\mathbb{B}^3_+, a, \mathbb{S}^2_+)$ induces the embedding $\mathbf{H}_{2n} \hookrightarrow \mathbf{B}_{2n}$ of the Hilden group in the braid group. We define the pure Hilden group to be the intersection of the Hilden group and the pure braid group.

$$\mathbf{PH}_{2n} = \mathbf{P}_{2n} \cap \mathbf{H}_{2n}$$

Generators for the corresponding subgroup of the braid group of the sphere were found by Hilden [4] and a finite presentation for the Hilden group was calculated independently by the Tawn [6, 7] and Brendle and Hatcher [2].

In [6, 7] we define an action of the Hilden group on a cellular complex and then use the method of Hatcher and Thurston [3], Wajnryb [8, 9, 10], etc., to compute a presentation of the group from this action. In this paper, we will use the same method with the same complex and action to prove the following theorem.

Theorem 1. *The pure Hilden group has a finite presentation with generating set S and relations R, $\mathbf{PH}_{2n} = \langle S \mid R\rangle$, where S and R are as follows.*

$$S = \left\{p_{\{i,j\}},\ x_{\{i,j\}},\ y_{\{i,j\}},\ t_k \mid 1 \le i < j \le n, 1 \le k \le n\right\}$$

To simplify notation we will write $p_{ij} = p_{\{i,j\}}$, $x_{ij} = x_{\{i,j\}}$ and $y_{ij} = y_{\{i,j\}}$, so, for example, $p_{ij} = p_{ji}$. The generators p_{ij}, x_{ij}, y_{ij} and t_k are the following elements of

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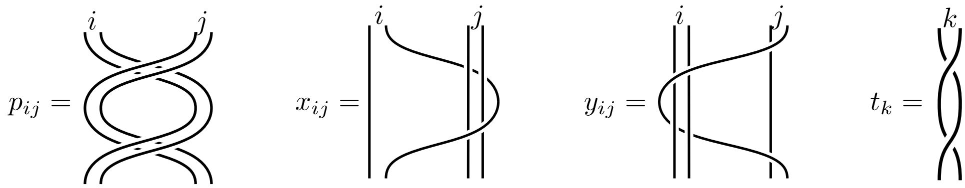
PH$_{2n}$. *Here all of the remaining strings lie behind those shown.*



The set R of relations is as follows.

(C-pt)	$p_{ij}\, t_k = t_k\, p_{ij}$	
(C-tt)	$t_i\, t_j = t_j\, t_i$	
(C-xt)	$x_{ij}\, t_k = t_k\, x_{ij}$	$i < j \qquad k \neq i$
(C-yt)	$y_{ij}\, t_k = t_k\, y_{ij}$	$i < j \qquad k \neq j$
(C1)	$\alpha_{ij}\, \beta_{kl} = \beta_{kl}\, \alpha_{ij}$	$\alpha, \beta \in \{p, x, y\}$ (i, j, k, l) *cyclically ordered*
(C2)	$\alpha_{ij}\ \beta_{ik}\, \gamma_{jk} = \beta_{ik}\, \gamma_{jk}\ \alpha_{ij}$	(i, j, k) *cyclically ordered* (α, β, γ) *as in Table 1*
(C3)	$\alpha_{ik}\ p_{jk}\, \beta_{jl}\, p_{jk}^{-1} = p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\ \alpha_{ik}$	$\alpha, \beta \in \{p, x, y\}$ (i, j, k, l) *cyclically ordered*
(C-xpt)	$x_{ij}\ p_{ij}\, t_i = p_{ij}\, t_i\ x_{ij}$	$i < j$
(C-ypt)	$y_{ij}\ p_{ij}\, t_j = p_{ij}\, t_j\ y_{ij}$	$i < j$

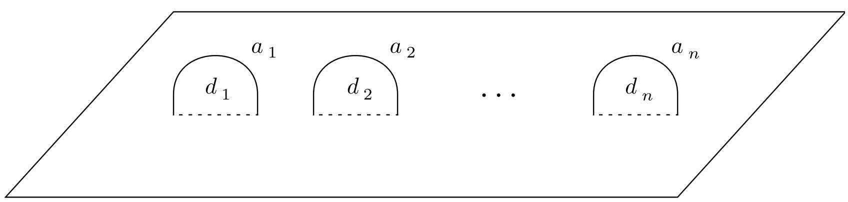


FIGURE 1. The arcs a_i and discs d_i.

i < j < k	(p,p,p)	(p,y,y)	(x,p,p)	(x,x,p)
	(x,y,y)	(y,p,p)	(y,p,x)	(y,y,y)
j < k < i	(p,p,p)	(p,x,y)	(x,p,p)	(x,p,x)
	(x,x,y)	(y,p,p)	(y,x,y)	(y,y,p)
k < i < j	(p,p,p)	(p,x,x)	(x,p,p)	(x,x,x)
	(x,y,p)	(y,p,p)	(y,p,y)	(y,x,x)

TABLE 1. The values of (α, β, γ) for (C2)

All of the relations are asserting that certain pairs of elements commute. The first four relations are the obvious ways in which a t can commute with the other generators. The relations (C1)–(C3) are analogous to the relations in the pure braid group. In fact, Table 1 lists all possible triples for which (C2) holds, these were found using the MAGMA computational algebra system [1]. To see why (C-xpt) and (C-ypt) hold it is easiest to consider the element $p_{ij}\,t_i\,t_j$. If you ignore all the other strings this is a full twist of the ith and jth pairs of strings and so clearly it commutes with x_{ij} and y_{ij}.

We refer the reader to [6, 7] for the definition of the cut–system complex and the method for computing the group presentation. In particular we will use the complex $\mathbf{X}_n$ with its associated $\mathbf{H}_{2n}$ action and basepoint $v_0 = \langle d_1, d_2, \ldots, d_n\rangle$. From the method we will use the notion of an h-product, the elements $\{r_\lambda\}$ and the notation for the generator sets S_0 and S_1 and the sets of relations R_0, R_1, R_2 and R_3.

In order to use the method to calculate a presentation, we need to show that the action on $\mathbf{X}_n$ is transitive on the vertex set and find edge and face orbit representatives.

Theorem 2. *The action of* $\mathbf{PH}_{2n}$ *on* $\mathbf{X}_n^0$ *is transitive.*

Proof. This follows from the proof that the action of $\mathbf{H}_{2n}$ on $\mathbf{X}_n^0$ is transitive given in [6]. All that is needed is to note that the constructed braids do not permute the punctures. □

2. Vertex stabilizer

Proposition 3. *The stabilizer of the vertex* v_0 *is the framed pure braid group* $\mathbf{FP}_n$ *and so is isomorphic to* $\mathbf{P}_n \times \mathbb{Z}^n$.

Proof. If we restrict our attention to $\mathbb{B}^2$, elements of $\mathbf{PH}_{2n}$ can be thought of as motions of the end points of the a_i. For elements of the stabilizer of v_0 this motion moves the line segments $d_i \cap \mathbb{B}^2$. So this is the fundamental group of configurations of n ordered line segments in the plain, the framed pure braid group. □

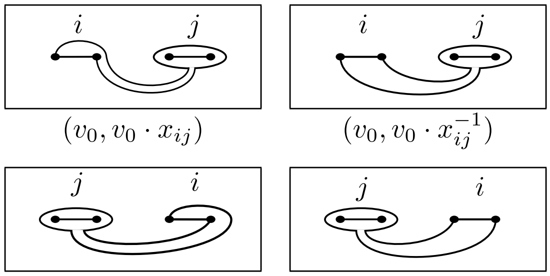
The pure braid group has a presentation with generators p_{ij} and relations (C1), (C2) and (C3) (with $\alpha = \beta = \gamma = p$). See, for example, Margalit and McCammond [5].

From this, we see that the vertex stabilizer is generated by the p_{ij} and t_k, that all relations between these elements follow from (C-pt), (C-tt), (C1), (C2) and (C3), and hence the R_0 relations are included in R.

3. Edge orbits

Let E denote the set of all oriented edges that start at v_0 the basepoint of $\mathbf{X}_n$. We will now find a representative of each orbit of the $\mathbf{FP}_n$ action on E, thus giving a set of $\mathbf{PH}_{2n}$ edge orbit representatives. Given an edge $(v_0, v) \in E$, because $v = \langle D_1, D_2, \ldots, D_n\rangle$ differs from v_0 by a simple move, there exists a unique i such that $D_i \neq d_i$.

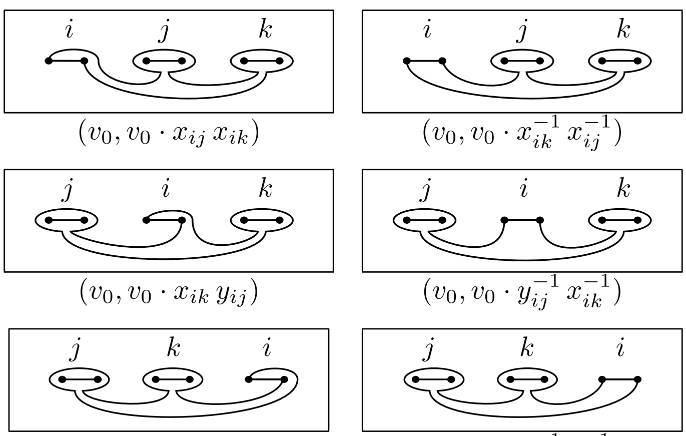
If the edge is of length one then there is a unique d_j under $D_i \cup d_i$. All of the remaining discs, d_k for $k \neq i, j$, can be moved by an element of $\mathbf{FP}_n$ away from $D_i \cup d_i$ and then back from behind to their original positions. After applying t_i^p for some p we have one of the following possibilities, each of which lie in a different orbit.



$(v_0, v_0 \cdot x_{ij})$ $(v_0, v_0 \cdot x_{ij}^{-1})$ for $i < j$

$(v_0, v_0 \cdot y_{ij})$ $(v_0, v_0 \cdot y_{ij}^{-1})$ for $j < i$

Similarly, if the edge is of length two then there exists two discs d_j and d_k under $d_i \cup D_i$. We may assume that $j < k$. As in the previous case there is an element of $\mathbf{FP}_n$ which takes (v_0, v) to one of the following possibilities, each of which lie in different orbits. There are three possible positions for d_j and d_k, either both lie to the right of d_i, there is one on either side, or they are both to the left.



$(v_0, v_0 \cdot x_{ij}\, x_{ik})$ $(v_0, v_0 \cdot x_{ik}^{-1}\, x_{ij}^{-1})$ for $i < j < k$

$(v_0, v_0 \cdot x_{ik}\, y_{ij})$ $(v_0, v_0 \cdot y_{ij}^{-1}\, x_{ik}^{-1})$ for $j < i < k$

$(v_0, v_0 \cdot y_{ij}\, y_{ik})$ $(v_0, v_0 \cdot y_{ik}^{-1}\, y_{ij}^{-1})$ for $j < k < i$

Proposition 4. *The pure Hilden group* $\mathbf{PH}_{2n}$ *is generated by* p_{ij}, t_i, x_{ij} *and* y_{ij}.

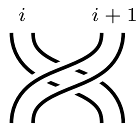
Proof. The group $\mathbf{PH}_{2n}$ is generated by the generators of the vertex stabilizer and the set $\{r_\lambda\}$. We have that

$$\{r_\lambda\} = \left\{ \begin{array}{cc} x_{ij}, & x_{ij}^{-1} \\ y_{ij}, & y_{ij}^{-1} \end{array} \;\middle|\; i < j \right\} \cup \left\{ \begin{array}{cc} x_{ij}\, x_{ik}, & x_{ik}^{-1}\, x_{ij}^{-1} \\ x_{jk}\, y_{ij}, & y_{ij}^{-1}\, x_{jk}^{-1} \\ y_{ik}\, y_{jk}, & y_{jk}^{-1}\, y_{ik}^{-1} \end{array} \;\middle|\; i < j < k \right\}$$

and so all of these generators either are contained in S or can be written in terms of the elements of S. □

4. Action of the framed braid group

We have an embedding of the framed braid group on n strings $\mathbf{FB}_n$ in the braid group on $2n$ strings given as follows.

$\sigma_i =$ 

$\tau_j =$ 

This makes $\mathbf{FB}_n$ a subgroup of $\mathbf{H}_{2n}$. It is clear that conjugation by elements of $\mathbf{FB}_n$ preserves the pure Hilden group and hence we have a left action of $\mathbf{FB}_n$ on $\mathbf{PH}_{2n}$. In fact, this action can be defined on the level of reduced words as well. In other words, we have an action of $F\langle \sigma_i, \tau_j\rangle$, the free group on the letters σ_i and τ_j, on $F\langle p_{ij}, x_{ij}, y_{ij}, t_k\rangle$, the free group on the letters p_{ij}, x_{ij}, y_{ij}, t_k.

In Section 8, we will construct this homomorphism

$$\Phi\colon F\langle \sigma_i, \tau_j\rangle \to \mathrm{Aut}(F\langle p_{ij}, x_{ij}, y_{ij}, t_k\rangle),$$

which we will write as $g \mapsto \Phi_g$, and then show that it satisfies the following properties. We will write $x =_{\mathbf{B}_{2n}} y$ when the words x and y have the same image in the braid group and $x =_R y$ when they have the same image in $\langle S \mid R\rangle$.

For any word $g \in F\langle \sigma_i, \tau_j\rangle$

(A) the automorphism Φ_g acts by conjugation at the level of braids,

$$\Phi_g(x) =_{\mathbf{B}_{2n}} g\,x\,g^{-1} \text{ for each } x \in F\langle p_{ij}, x_{ij}, y_{ij}, t_k\rangle$$

(B) the automorphism Φ_g preserves the subgroup generated by the p_{ij} and t_k,

$$\Phi_g(h) \in F\langle p_{ij}, t_k\rangle \text{ for each } h \in F\langle p_{ij}, t_k\rangle$$

(C) for each r_λ there exists $h_1, h_2 \in F\langle p_{ij}, t_k\rangle$ and $r_{\lambda'}$ such that $\Phi_g(r_\lambda) =_R h_1 r_{\lambda'} h_2$,

(D) if $x =_R y$ then $\Phi_g(x) =_R \Phi_g(y)$.

We will now assume the existence of such a Φ and use it to show that R_1, R_2 and R_3 relations follow from those in R.

5. The R_1 relations

The R_1 relations consist of a relation of the form $r_\lambda\, t r_\lambda^{-1} = h$ for each edge orbit representative $(v_0, v_0 \cdot r_\lambda)$, for each t in a generating set of the stabilizer of this edge and for some word h in $\mathbf{FB}_n$.

Proposition 5. *The stabilizer of the edge $(v_0, v_0 \cdot x_{12})$ is generated as follows.*

$$\mathrm{Stab}(v_0, v_0 \cdot x_{12}) = \left\langle \begin{matrix} p_{ij} & \text{for } i, j > 2, & t_k & \text{for } k > 1, \\ p_{12}t_1, & & p_{1j}\ p_{2j} & \text{for } j > 2 \end{matrix} \right\rangle$$

Proof. The stabilizer can be viewed as the mapping class group of the disc fixing pointwise both its boundary and the arcs shown in Figure 2. The two arcs connecting the first two punctures form a loop l. As this loop is fixed by elements of the stabilizer, this allows us to split the group into the product of two (boundary fixing) mapping class groups. One corresponding to the inside of l and the other to the outside.

For the inside, we get the mapping class group of the annulus, which is cyclic and generated by t_2.

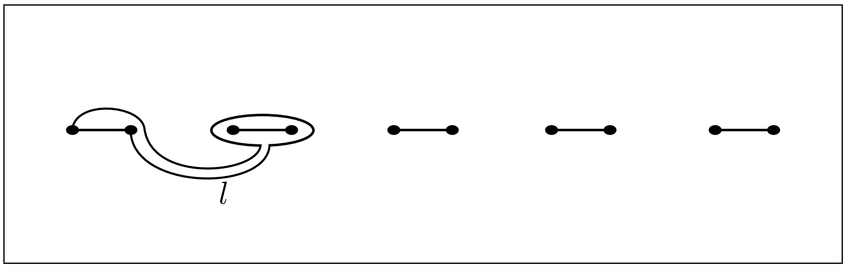


FIGURE 2. The edge $(v_0, v_0 \cdot x_{12})$.

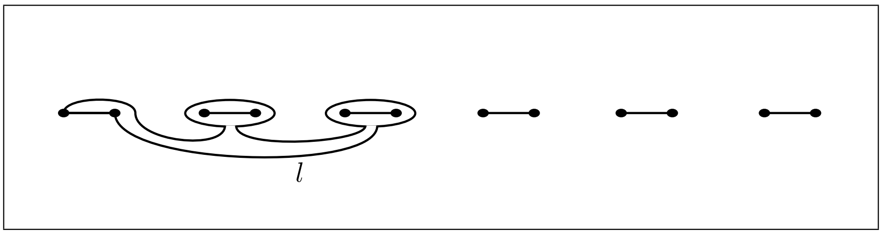


FIGURE 3. The edge $(v_0, v_0 \cdot x_{12}\, x_{13})$.

For the outside, we get the mapping class group of a disc with $n-1$ subdiscs removed. This is $\mathbf{FP}_{n-1}$ the framed pure braid group on $n-1$ strings. The framing of the subdisc bounded by the loop l is generated by $p_{12}t_1t_2$ and for the other subdiscs by t_k for $k > 2$. The braidings of the subdiscs are generated by $p_{1j}p_{2j}$ for $j > 2$ and p_{ij} for $i, j > 2$. □

So the R_1 relations can be chosen as follows.

$$x_{12}\ p_{ij}\ x_{12}^{-1} = p_{ij} \qquad \text{for } 2 < i < j \tag{5.1}$$

$$x_{12}\ t_k\ x_{12}^{-1} = t_k \qquad \text{for } k > 1 \tag{5.2}$$

$$x_{12}\ p_{12}\, t_1\ x_{12}^{-1} = p_{12}\, t_1 \tag{5.3}$$

$$x_{12}\ p_{1j}\, p_{2j}\ x_{12}^{-1} = p_{1j}\, p_{2j} \qquad \text{for } j > 2 \tag{5.4}$$

Relation (5.1) follows from (C1), relation (5.2) follows from (C-xt), relation (5.3) follows from (C-xpt) and relation (5.4) follows from (C2).

Proposition 6. *The stabilizer of the edge $(v_0, v_0 \cdot x_{12}\, x_{13})$ is generated as follows.*

$$\mathrm{Stab}(v_0, v_0 \cdot x_{12}\, x_{13}) = \left\langle \begin{matrix} p_{23}, & p_{12}\, p_{13}\, t_1, & p_{ij} & \text{for } i, j > 3, \\ t_k & \text{for } k > 1\ \ , & p_{1j}\, p_{2j}\, p_{3j} & \text{for } j > 3 \end{matrix} \right\rangle$$

Proof. As with the previous proposition, the stabilizer can be viewed as the mapping class group of a disc fixing pointwise both its boundary and the arcs shown in Figure 3. Again, we have a loop l formed from the two arcs joining the first two punctures, which allows us to split the group into a product of two mapping class groups.

Inside l we get the mapping class group of a disc with two subdiscs removed. This is $\mathbf{FP}_2$ the framed pure braid group on two strings. The framing of the strings is generated by t_2 and t_3 and the braiding is generated by p_{23}.

Outside l we get the mapping class group of a disc with $n-2$ subdiscs removed, i.e. $\mathbf{FP}_{n-2}$. The framing of the subdisc bounded by l is generated by $p_{12}\,p_{13}\,t_1\;t_2\;t_3$ and the framing of the other subdiscs is given by t_k for $k>3$. The braidings of the subdiscs are generated by $p_{1j}\,p_{2j}\,p_{3j}$ for $j>3$ and p_{ij} for $i,j>3$. □

Hence the R_1 relations can be chosen as follows.

$$x_{12}\,x_{13}\;p_{23}\;(x_{12}\,x_{13})^{-1}=p_{23} \tag{5.5}$$

$$x_{12}\,x_{13}\;p_{ij}\;(x_{12}\,x_{13})^{-1}=p_{ij} \qquad \text{for } i,j>3 \tag{5.6}$$

$$x_{12}\,x_{13}\;t_k\;(x_{12}\,x_{13})^{-1}=t_k \qquad \text{for } k>1 \tag{5.7}$$

$$x_{12}\,x_{13}\;p_{12}\,p_{13}\,t_1\;(x_{12}\,x_{13})^{-1}=p_{12}\,p_{13}\,t_1 \tag{5.8}$$

$$x_{12}\,x_{13}\;p_{1j}\,p_{2j}\,p_{3j}\;(x_{12}\,x_{13})^{-1}=p_{1j}\,p_{2j}\,p_{3j} \qquad \text{for } j>3 \tag{5.9}$$

Relation (5.5) follows from (C2), relation (5.6) follows from two applications of (C1), relation (5.7) follows from two applications of (C-xt). Relation (5.8) follows from the following.

$$\begin{aligned}
&x_{12}\,x_{13}\,\underline{p_{12}\,p_{13}}\,t_1 && \text{(C2)}\\
&=x_{12}\,x_{13}\,p_{13}\,p_{23}\,p_{12}\,p_{23}^{-1}\,\underline{t_1} && \text{(C-}pt)^3\\
&=x_{12}\,\underline{x_{13}\,p_{13}\,t_1}\,p_{23}\,p_{12}\,p_{23}^{-1} && \text{(C-}xpt)\\
&=x_{12}\,p_{13}\,t_1\,\underline{x_{13}\,p_{23}\,p_{12}}\,p_{23}^{-1} && \text{(C2)}\\
&=x_{12}\,p_{13}\,\underline{t_1}\,p_{23}\,p_{12}\,x_{13}\,p_{23}^{-1} && \text{(C-}pt)^2\\
&=\underline{x_{12}\,p_{13}\,p_{23}}\,p_{12}\,t_1\,x_{13}\,p_{23}^{-1} && \text{(C2)}\\
&=p_{13}\,p_{23}\,\underline{x_{12}\,p_{12}\,t_1}\,x_{13}\,p_{23}^{-1} && \text{(C-}xpt)\\
&=p_{13}\,p_{23}\,p_{12}\,t_1\,\underline{x_{12}\,x_{13}\,p_{23}^{-1}} && \text{(C2)}\\
&=p_{13}\,p_{23}\,p_{12}\,\underline{t_1\,p_{23}^{-1}}\,x_{12}\,x_{13} && \text{(C-}pt)\\
&=\underline{p_{13}\,p_{23}\,p_{12}\,p_{23}^{-1}}\,t_1\,x_{12}\,x_{13} && \text{(C2)}\\
&=p_{12}\,p_{13}\,t_1\,x_{12}\,x_{13}
\end{aligned}$$

Finally, (5.9) follows from the following.

$$\begin{aligned}
&\underline{x_{13}\,p_{1k}}\,p_{2k}\,p_{3k} && \text{(C2)}\\
&=p_{1k}\,p_{3k}\,x_{13}\,\underline{p_{3k}^{-1}\,p_{2k}\,p_{3k}} && \text{(C2)}\\
&=p_{1k}\,p_{3k}\,\underline{x_{13}\,p_{23}\,p_{2k}\,p_{23}^{-1}} && \text{(C3)}\\
&=p_{1k}\,\underline{p_{3k}\,p_{23}\,p_{2k}\,p_{23}^{-1}}\,x_{13} && \text{(C2)}\\
&=p_{1k}\,p_{2k}\,p_{3k}\,x_{13}
\end{aligned}$$

$$\underline{x_{12}\,p_{1k}\,p_{2k}}\,p_{3k}\overset{\text{(C2)}}{=}p_{1k}\,p_{2k}\,\underline{x_{12}\,p_{3k}}\overset{\text{(C1)}}{=}p_{1k}\,p_{2k}\,p_{3k}\,x_{12}$$

Now consider the edge orbit representative $(v_0, v_0 \cdot r_\lambda)$ for $r_\lambda \neq x_{12}$ or $x_{12}\, x_{13}$. There exist some $g \in \mathbf{FB}_n$ such that $(v_0, v_0 \cdot r_1) \cdot g = (v_0, v_0 \cdot r_\lambda)$, where $r_1 = x_{12}$ or $x_{12}\, x_{13}$. By property (A) of Φ, we have $\Phi_{g^{-1}}(r_1) =_{\mathbf{B}_{2n}} g^{-1} r_1 g$ and by property (C) there exists words $h_1, h_2 \in \mathbf{FP}_n$ and some $r_{\lambda'}$ such that

$$\Phi_{g^{-1}}(r_1) =_R h_1\, r_{\lambda'}\, h_2. \tag{5.10}$$

Combining these we see that $v_0 \cdot r_1\, g = v_0 \cdot r_{\lambda'}\, h_2$ and hence that $\lambda = \lambda'$ and $h_2 \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$.

Let T be the choice of generators for $\mathrm{Stab}(v_0, v_0 \cdot r_1)$ chosen above. So for all $t \in T$ there exists $h \in \mathbf{FP}_n$ such that $r_1\, t\, r_1^{-1} =_R h$. So by property (D) we have

$$\Phi_{g^{-1}}(r_1\, t\, r_1^{-1}) =_R \Phi_{g^{-1}}(h). \tag{5.11}$$

Property (B) implies that $\Phi_{g^{-1}}(t) \in \mathbf{FP}_n$ and $\Phi_{g^{-1}}(h) \in \mathbf{FP}_n$. Combining (5.10) and (5.11) we get

$$h_1\, r_\lambda\, h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1}\, r_\lambda^{-1}\, h_1^{-1} =_R \Phi_{g^{-1}}(h)$$

and so $h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1} \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$.

Claim. *The set* $\{h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1} \mid t \in T\}$ *generates* $\mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$.

Proof. As $h_2 \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$ the set $\{h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1} \mid t \in T\}$ generates $\mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$ if and only if the set $\{\Phi_{g^{-1}}(t) \mid t \in T\}$ generates $\mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$. This is equivalent to saying that for any $s \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$ we can find $t_1, \ldots, t_k \in T$ such that $s = \Phi_{g^{-1}}(t_1 \ldots t_k)$, in other words that $\Phi_g(s) \in \mathrm{Stab}(v_0, v_0 \cdot r_1)$. Now

$$(v_0 \cdot r_1) \cdot \Phi_g(s) = v_0 \cdot r_1\, g\, s\, g^{-1} = v_0 \cdot r_\lambda\, s\, g^{-1} = v_0 \cdot r_\lambda\, g^{-1} = v_0 \cdot r_1.$$

Therefore the claim holds. ☐

So for our R_1 relation we can choose

$$r_\lambda\ h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1}\ r_\lambda^{-1} = h_1^{-1}\, \Phi_{g^{-1}}(h)\, h_1$$

and hence we can choose our R_1 relations so that they all follow from R.

6. The R_2 relations

The R_2 relations consist of a relation of the form $r_{\lambda'}\, h\, r_\lambda = h'$ for each edge orbit representative, where the left-hand side (LHS) is an h-product for the path $(v_0, v_0 \cdot r_\lambda, v_0)$ and $h' \in \mathbf{FB}_n$. For each edge $(v_0, v_0 \cdot r_\lambda)$ the edge $(v_0, v_0 \cdot r_\lambda^{-1})$ is in a different orbit. Our choice of r_λ mean that for all λ there exists λ' such that $r_\lambda^{-1} = r_{\lambda'}$. This means that for all the R_2 relations we can choose $r_\lambda^{-1}\, r_\lambda = 1$, i.e. they are all trivial.

7. The R_3 relations

The R_3 relations consist of a relation of the form $r_{\lambda_k} h_k\, \ldots\, r_{\lambda_1} h_1 = h$ for each face orbit representative, where the LHS is an h-product that represents the boundary of the face and $h \in \mathbf{FP}_n$. As with the R_1 relations, we will calculate the relations for some specific orbits first then use Φ for the general case. There are three types of faces: triangular, non-nested rectangular and nested rectangular.

We will start with the triangular face $(v_0, v_0 \cdot x_{12}\, x_{13}, v_0 \cdot x_{13}, v_0)$. An h-product for this path is $x_{13}^{-1}\ x_{12}^{-1}\ (x_{12}\, x_{13})$. So the R_3 relations is $x_{13}^{-1}\ x_{12}^{-1}\ (x_{12}\, x_{13}) = 1$ and so it is trivial.

Next consider the non-nested rectangular face $(v_0,\ v_0 \cdot x_{12},\ v_0 \cdot x_{34}\, x_{12},\ v_0 \cdot x_{34},\ v_0)$. An h-product that represents this path is $x_{34}^{-1}\ x_{12}^{-1}\ x_{34}\ x_{12}$. So the R_3 relations is $x_{34}^{-1}\ x_{12}^{-1}\ x_{34}\ x_{12} = 1$, which follows from (C1).

Now consider the nested rectangular face $(v_0,\ v_0 \cdot x_{23},\ v_0 \cdot x_{12}\, x_{13}\, x_{23},\ v_0 \cdot x_{12}\, x_{13}, v_0)$. An h-product that represents this path is $(x_{12}\, x_{13})^{-1} x_{23}^{-1}\, (x_{12}\, x_{13})\, x_{23}$. So the R_3 relations is $(x_{12}\, x_{13})^{-1}\ x_{23}^{-1}\, (x_{12}\, x_{13})\, x_{23} = 1$, which follows from (C2).

Given any other face orbit representative $(v_0 = u_0, u_1, \ldots, u_k = v_0)$ then by the classification of face orbits given in [6] there exist some $g \in \mathbf{FB}_n$ such that $(u_0, u_1, \ldots, u_k) = (v_0, v_1, \ldots, v_k) \cdot g$ where $(v_0, v_1, \ldots, v_k)$ is the boundary of one of the three faces whose R_3 relations we calculated above. Suppose the relation from $(v_0, v_1, \ldots, v_k)$ is the following.

$$r_{\lambda_k} h_k \ldots r_{\lambda_1} h_1 = h$$

By property (C), for each r_{λ_i} there exists $h_{i1}, h_{i2} \in \mathbf{FP}_n$ and $r_{\lambda_i'}$ such that

$$\Phi_{g^{-1}}(r_{\lambda_i}) =_R h_{i1}\, r_{\lambda_i'}\, h_{i2}.$$

Claim. *The following h-product represents the path* $(u_0, u_1, \ldots, u_k)$.

$$r_{\lambda_k'}\, h_{k2}\, \Phi_{g^{-1}}(h_k)\, h_{(k-1)1}\ \ldots\ r_{\lambda_1'}\, h_{11}\, \Phi_{g^{-1}}(h_1)$$

Proof. The ith vertex of the path associated to the h-product is given as follows.

$$\begin{aligned}
&v_0 \cdot r_{\lambda_i'}\, h_{i2}\, \Phi_{g^{-1}}(h_i)\, h_{(i-1)1}\ \ldots\ r_{\lambda_1'}\, h_{11}\, \Phi_{g^{-1}}(h_1) \\
&\quad = v_0 \cdot \Phi_{g^{-1}}(r_{\lambda_i}\, h_i \ldots r_{\lambda_1}\, h_1) \\
&\quad = v_0 \cdot r_{\lambda_i}\, h_i \ldots r_{\lambda_1}\, h_1\, g \\
&\quad = v_i \cdot g \\
&\quad = u_i
\end{aligned}$$

□

Therefore, for our R_3 relation we may choose

$$r_{\lambda'_k}\, h_{k2}\, \Phi_{g^{-1}}(h_k)\, h_{(k-1)1} \cdots r_{\lambda'_1}\, h_{11}\, \Phi_{g^{-1}}(h_1) = h_{k1}^{-1}\, \Phi_{g^{-1}}(h),$$

which follows from R by property (D).

8. Construction and properties of Φ

All that remains to prove Theorem 1 is to construct Φ and show that it satisfies properties (A)–(D).

Define Φ, the action of $F\langle \sigma_i,\ \tau_j\rangle$ on $F\langle\, p_{ij},\ x_{ij},\ y_{ij},\ t_k\rangle$, as follows. For $\alpha \in \{p, x, y\}$

$$\begin{aligned}
\Phi_{\sigma_i}(\alpha_{kl}) &= \alpha_{kl} && \text{for } i \neq k-1, k, l-1, l\\
\Phi_{\sigma_i}(\alpha_{ij}) &= \alpha_{i+1,j} && \text{for } i+1 < j\\
\Phi_{\sigma_i}(\alpha_{i+1,j}) &= p_{i,i+1}\, \alpha_{ij}\, p_{i,i+1}^{-1} && \text{for } i+1 < j\\
\Phi_{\sigma_j}(\alpha_{i,j+1}) &= p_{j,j+1}\, \alpha_{ij}\, p_{j,j+1}^{-1} && \text{for } i+1 < j\\
\Phi_{\sigma_j}(\alpha_{ij}) &= \alpha_{i,j+1} && \text{for } i+1 < j
\end{aligned}$$

$$\Phi_{\sigma_i}(p_{i,i+1}) = p_{i,i+1} \quad \Phi_{\sigma_i}(x_{i,i+1}) = t_{i+1}^{-1}\, y_{i,i+1}\, t_{i+1} \quad \Phi_{\sigma_i}(y_{i,i+1}) = x_{i,i+1}$$

$$\Phi_{\sigma_i}(t_j) = \begin{cases} t_j & \text{if } j \neq i,\ i+1\\ t_{j+1} & \text{if } j = i\\ t_i & \text{if } j = i+1\end{cases}$$

$$\Phi_{\tau_i}(p_{kl}) = p_{kl}$$

$$\Phi_{\tau_i}(x_{kl}) = \begin{cases} x_{kl} & \text{if } i \neq k\\ x_{kl}^{-1}\, p_{kl} & \text{if } i = k\end{cases} \qquad \text{for } k < l$$

$$\Phi_{\tau_i}(y_{kl}) = \begin{cases} y_{kl} & \text{if } i \neq l\\ y_{kl}^{-1}\, p_{kl} & \text{if } i = l\end{cases} \qquad \text{for } k < l$$

$$\Phi_{\tau_i}(t_j) = t_j$$

Proposition 7. *The map Φ is a well-defined action of $F\langle\, \tau_i,\ \sigma_i\,\rangle$ on $F\langle\, p_{ij},\ t_i, x_{ij},\ y_{ij}\rangle$.*

Proof. All that needs to be checked is that Φ_{σ_i} and Φ_{τ_i} are invertible. The inverses are as follows.

$$\begin{aligned}
\Phi_{\sigma_i^{-1}}(\alpha_{kl}) &= \alpha_{kl} && \text{for } i \neq k-1, k, l-1, l\\
\Phi_{\sigma_i^{-1}}(\alpha_{ij}) &= p_{i,i+1}^{-1}\, \alpha_{i+1,j}\, p_{i,i+1} && \text{for } i+1 < j\\
\Phi_{\sigma_i^{-1}}(\alpha_{i+1,j}) &= \alpha_{ij} && \text{for } i+1 < j\\
\Phi_{\sigma_j^{-1}}(\alpha_{i,j+1}) &= \alpha_{ij} && \text{for } i+1 < j\\
\Phi_{\sigma_j^{-1}}(\alpha_{ij}) &= p_{j,j+1}^{-1}\, \alpha_{i,j+1}\, p_{j,j+1} && \text{for } i+1 < j
\end{aligned}$$

$$\Phi_{\sigma_i^{-1}}(p_{i,i+1}) = p_{i,i+1} \quad \Phi_{\sigma_i^{-1}}(x_{i,i+1}) = y_{i,i+1} \quad \Phi_{\sigma_i^{-1}}(y_{i,i+1}) = t_i\, x_{i,i+1}\, t_i^{-1}$$

$$\Phi_{\sigma_i^{-1}}(t_j) = \begin{cases} t_j & \text{if } j \neq i,\, i+1 \\ t_{j+1} & \text{if } j = i \\ t_{j-1} & \text{if } j = i+1 \end{cases}$$

$$\Phi_{\tau_i^{-1}}(p_{kl}) = p_{kl}$$

$$\Phi_{\tau_i^{-1}}(x_{kl}) = \begin{cases} x_{kl} & \text{if } i \neq k \\ p_{kl}\, x_{kl}^{-1} & \text{if } i = k \end{cases} \qquad \text{for } k < l$$

$$\Phi_{\tau_i^{-1}}(y_{kl}) = \begin{cases} y_{kl} & \text{if } i \neq l \\ p_{kl}\, y_{kl}^{-1} & \text{if } i = l \end{cases} \qquad \text{for } k < l$$

$$\Phi_{\tau_i^{-1}}(t_j) = t_j$$

□

We will need the following lemma.

Lemma 8. *For $x \in F\langle\, p_{ij},\, t_i,\, x_{ij},\, y_{ij}\rangle$ we have*

$$\Phi_{\sigma_m^{-2}}(x) =_R p_{m,m+1}^{-1}\, x\, p_{m,m+1} \qquad\qquad \Phi_{\tau_m^{-2}}(x) =_R t_m^{-1}\, x\, t_m.$$

It is easy to check the Φ satisfies property (A), i.e. that for every word $g \in F\langle\sigma_i,\, \tau_j\rangle$ and for each $x \in F\langle\, p_{ij},\, t_i,\, x_{ij},\, y_{ij}\rangle$ we have that $\Phi_g(x) = g\, x\, g^{-1}$ as braids. It is also clear that Φ satisfies property (B). That is that for any word $g \in F\langle\sigma_i,\, \tau_j\rangle$ and for any word $h \in F\langle\, p_{ij},\, t_k\rangle$ we have $\Phi_g(h) \in F\langle\, p_{ij},\, t_k\rangle$.

Proposition 9. *The map Φ satisfies property (C), i.e. for any word $g \in F\langle\sigma_i,\, \tau_j\rangle$ and any r_λ we have a relation $\Phi_g(r_\lambda) = h_1 r_{\lambda'} h_2$ for some $h_1, h_2 \in F\langle\, p_{ij},\, t_k\rangle$ and some $r_{\lambda'}$ that can be deduced from the relations in R.*

Proof. First note that for each word h in $F\langle\, p_{ij},\, t_k\rangle$, by property (B), the map Φ_g takes h to another word in $F\langle\, p_{ij},\, t_k\rangle$. Therefore, it suffices to check Φ_g for $g = \tau_m$, σ_m, τ_m^{-2} and σ_m^{-2}. By Lemma 8, property (C) is satisfied for $g = \tau_m^{-2}$ and σ_m^{-2}.

For $r_\lambda = x_{ij}, x_{ij}^{-1}, y_{ij}, y_{ij}^{-1}$ this follows immediately from the definition of Φ given above.

Now consider $\Phi_{\sigma_m}(r_\lambda)$ for $r_\lambda = x_{ij}\, x_{ik},\ x_{jk}\, y_{ij}$ or $y_{ik}\, y_{jk}$. The only cases when $\Phi_{\sigma_m}(r_\lambda) \neq r_\lambda$ are $m = i-1$, $m = i$ and $j = i+1$, $m = i$ and $j > i+1$, $m = j-1$ and $i < j-1$, $m = j$ and $k = j+1$, $m = j$ and $k > j+1$, $m = k-1$ and $j < k-1$, and $m = k$.

$m = i-1$

$$\Phi_{\sigma_{i-1}}(x_{ij}\, x_{ik}) = p_{i-1,i}\ x_{i-1,j}\, x_{i-1,k}\ p_{i-1,i}^{-1}$$

$$\begin{aligned} \Phi_{\sigma_{i-1}}(x_{jk}\, y_{ij}) &= \underline{x_{jk}\, p_{i-1,i}}\, y_{i-1,j}\, p_{i-1,i}^{-1} &\text{(C1)} \\ &= p_{i-1,i}\ x_{jk}\, y_{i-1,j}\ p_{i-1,i}^{-1} \end{aligned}$$

$$\begin{aligned} \Phi_{\sigma_{i-1}}(y_{ik}\, y_{jk}) &= p_{i-1,i}\, y_{i-1,k}\, \underline{p_{i-1,i}^{-1}\, y_{jk}} &\text{(C1)} \\ &= p_{i-1,i}\ y_{i-1,k}\, y_{jk}\ p_{i-1,i}^{-1} \end{aligned}$$

$m = i$ and $j = i + 1$

$$\begin{aligned}
\Phi_{\sigma_i}(x_{ij}\,x_{ik}) &= t_j^{-1}\,y_{ij}\,\underline{t_j\,x_{jk}} && \text{(C-}xpt\text{)}\\
&= t_j^{-1}\,\underline{y_{ij}\,p_{jk}^{-1}}\,x_{jk}\,p_{jk}t_j && \text{(C2)}\\
&= t_j^{-1}\,p_{jk}^{-1}\,\underline{p_{ik}^{-1}\,y_{ij}\,p_{ik}\,x_{jk}}\,p_{jk}\,t_j && \text{(C2)}\\
&= t_j^{-1}\,p_{jk}^{-1}\;x_{jk}\,y_{ij}\;p_{jk}\,t_j\\
\Phi_{\sigma_i}(x_{jk}\,y_{ij}) &= \underline{p_{ij}\,x_{ik}\,p_{ij}^{-1}}\,x_{ij} && \text{(C2)}\\
&= p_{jk}^{-1}\,\underline{x_{ik}\,p_{jk}\,x_{ij}} && \text{(C2)}\\
&= p_{jk}^{-1}\;x_{ij}\,x_{ik}\;p_{jk}\\
\Phi_{\sigma_i}(y_{ik}\,y_{jk}) &= \underline{y_{jk}\,p_{ij}\,y_{ik}\,p_{ij}^{-1}} && \text{(C2)}\\
&= y_{ik}\,y_{jk}
\end{aligned}$$

$m = i$ and $j > i + 1$

$$\begin{aligned}
\Phi_{\sigma_i}(x_{ij}\,x_{ik}) &= x_{i+1,j}\,x_{i+1,k}\\
\Phi_{\sigma_i}(x_{jk}\,y_{ij}) &= x_{jk}\,y_{i+1,j}\\
\Phi_{\sigma_i}(y_{ik}\,y_{jk}) &= y_{i+1,k}\,y_{jk}
\end{aligned}$$

$m = j - 1$ and $i < j - 1$

$$\begin{aligned}
\Phi_{\sigma_{j-1}}(x_{ij}\,x_{ik}) &= p_{j-1,j}\,x_{i,j-1}\,\underline{p_{j-1,j}^{-1}\,x_{ik}} && \text{(C1)}\\
&= p_{j-1,j}\;x_{i,j-1}\,x_{ik}\;p_{j-1,j}^{-1}\\
\Phi_{\sigma_{j-1}}(x_{jk}\,y_{ij}) &= p_{j-1,j}\;x_{j-1,k}\,y_{i,j-1}\;p_{j-1,j}^{-1}\\
\Phi_{\sigma_{j-1}}(y_{ik}\,y_{jk}) &= \underline{y_{ik}\,p_{j-1,j}}\,y_{j-1,k}\,p_{j-1,j}^{-1} && \text{(C1)}\\
&= p_{j-1,j}\;y_{ik}\,y_{j-1,k}\;p_{j-1,j}^{-1}
\end{aligned}$$

$m = j$ and $k = j + 1$

$$\begin{aligned}
\Phi_{\sigma_j}(x_{ij}\,x_{ik}) &= \underline{x_{ik}\,p_{jk}\,x_{ij}\,p_{jk}^{-1}} && \text{(C2)}\\
&= x_{ij}\,x_{ik}\\
\Phi_{\sigma_j}(x_{jk}\,y_{ij}) &= t_k^{-1}\,\underline{y_{jk}}\,t_k\,y_{ik} && \text{(C-}ypt\text{)}\\
&= \underline{t_k^{-1}\,p_{jk}\,t_k}\,y_{jk}\,\underline{t_k^{-1}\,p_{jk}^{-1}\,t_k}\,y_{ik} && \text{(C-}pt\text{)}^2\\
&= p_{jk}\,y_{jk}\,\underline{p_{jk}^{-1}\,y_{ik}} && \text{(C2)}\\
&= p_{jk}\,\underline{y_{jk}\,p_{ij}\,y_{ik}\,p_{ij}^{-1}}\,p_{jk}^{-1} && \text{(C2)}\\
&= p_{jk}\;y_{ik}\,y_{jk}\;p_{jk}^{-1}\\
\Phi_{\sigma_j}(y_{ik}\,y_{jk}) &= \underline{p_{jk}\,y_{ij}\,p_{jk}^{-1}}\,x_{jk} && \text{(C2)}\\
&= \underline{p_{ik}^{-1}\,y_{ij}\,p_{ik}\,x_{jk}} && \text{(C2)}\\
&= x_{jk}\,y_{ij}.
\end{aligned}$$

$m = j$ and $k > j + 1$

$$\begin{aligned}
\Phi_{\sigma_j}(x_{ij}\,x_{ik}) &= x_{i,j+1}\,x_{ik}\\
\Phi_{\sigma_j}(x_{jk}\,y_{ij}) &= x_{j+1,k}\,y_{i,j+1}\\
\Phi_{\sigma_j}(y_{ik}\,y_{jk}) &= y_{ik}\,y_{j+1,k}
\end{aligned}$$

$$m = k-1 \text{ and } j < k-1 \quad \Phi_{\sigma_{k-1}}(x_{ij}\, x_{ik}) = \underline{x_{ij}\, p_{k-1,k}}\, x_{i,k-1}\, p_{k-1,k}^{-1} \overset{}{} \quad (C1)$$

$$= p_{k-1,k}\; x_{ij}\, x_{i,k-1}\; p_{k-1,k}^{-1}$$

$$\Phi_{\sigma_{k-1}}(x_{jk}\, y_{ij}) = p_{k-1,k}\, x_{j,k-1}\, \underline{p_{k-1,k}^{-1}\, y_{ij}} \quad (C1)$$

$$= p_{k-1,k}\; x_{j,k-1}\, y_{ij}\; p_{k-1,k}^{-1}$$

$$\Phi_{\sigma_{k-1}}(y_{ik}\, y_{jk}) = p_{k-1,k}\; y_{i,k-1}\, y_{j,k-1}\; p_{k-1,k}^{-1}$$

$$m = k \quad \Phi_{\sigma_k}(x_{ij}\, x_{ik}) = x_{ij}\, x_{i,k+1},$$

$$\Phi_{\sigma_k}(x_{jk}\, y_{ij}) = x_{j,k+1}\, y_{ij},$$

$$\Phi_{\sigma_k}(y_{ik}\, y_{jk}) = y_{i,k+1}\, y_{j,k+1}.$$

For Φ_{τ_m} we only have three cases where $\Phi_{\tau_m}(r_\lambda) \neq r_\lambda$ these are when $m = i$ and $r_\lambda = x_{ij}\, x_{ik}$, $m = j$ and $r_\lambda = x_{jk}\, y_{ij}$, and $m = k$ and $r_\lambda = y_{ik}\, y_{jk}$.

$$\Phi_{\tau_i}(x_{ij}\, x_{ik}) = x_{ij}^{-1}\, \underline{p_{ij}\, x_{ik}^{-1}}\, p_{ik} \overset{(C2)}{=} \underline{x_{ij}^{-1}\, p_{jk}^{-1}\, x_{ik}^{-1}\, p_{jk}}\, p_{ij}\, p_{ik} \overset{(C2)}{=} x_{ik}^{-1}\, x_{ij}^{-1}\; p_{ij}\, p_{ik}$$

$$\Phi_{\tau_j}(x_{jk}\, y_{ij}) = x_{jk}^{-1}\, \underline{p_{jk}\, y_{ij}^{-1}}\, p_{ij} \overset{(C2)}{=} \underline{x_{jk}^{-1}\, p_{ik}^{-1}\, y_{ij}^{-1}\, p_{ik}}\, p_{jk}\, p_{ij} \overset{(C2)}{=} y_{ij}^{-1}\, x_{jk}^{-1}\; p_{jk}\, p_{ij}$$

$$\Phi_{\tau_k}(y_{ik}\, y_{jk}) = y_{ik}^{-1}\, \underline{p_{ik}\, y_{jk}^{-1}}\, p_{jk} \overset{(C2)}{=} \underline{y_{ik}^{-1}\, p_{ij}^{-1}\, y_{jk}^{-1}\, p_{ij}}\, p_{ik}\, p_{jk} \overset{(C2)}{=} y_{jk}^{-1}\, y_{ik}^{-1}\; p_{ik}\, p_{jk}$$

For $r_\lambda = x_{ik}^{-1} x_{ij}^{-1}$, $y_{ij}^{-1} x_{ik}^{-1}$ and $y_{jk}^{-1} y_{ik}^{-1}$ we have shown that for some $h_1, h_2 \in \mathbf{FP}_n$ and some $r_{\lambda'}^{-1}$ we have that $\Phi_g(r_\lambda^{-1}) =_R h_1\; r_{\lambda'}^{-1}\; h_2$. Hence, we have $\Phi_g(r_\lambda) =_R h_2^{-1}\; r_{\lambda'}\; h_1^{-1}$. □

Proposition 10. *The map Φ satisfies property (D). In other words, for any word $g \in F\langle \sigma_i, \tau_j\rangle$ and any relation $x =_R y$ we have that $\Phi_g(x) =_R \Phi_g(y)$.*

Proof. As in the proof of property (C), it suffices to show this for g in a monoidal generating set for $F\langle \sigma_i, \tau_j\rangle$. For $g = \sigma_i^{-2}$ and τ_j^{-2} this follows from Lemma 8, so it remains to show it for $g = \sigma_i$ and τ_j.

For any relation only involving p_{ij}'s and t_k's the image under Φ_g will still only involve p_{ij}'s and t_k's and hence, by Section 2, the new relation will follow from those in R.

We will now consider the action of Φ_{σ_q} and Φ_{τ_q} on each of the relations. For any relation $x =_R y$, we will say that the deduction of $\Phi_g(x) = \Phi_g(y)$ is trivial if $\Phi_g(x) = \Phi_g(y)$ is a relation in R of the same type.

Case 1. (C-xt) $\quad x_{ij}\, t_k = t_k\, x_{ij} \quad k \neq i,\ i < j$

First consider Φ_{σ_q}. Start with $q = 1$ and then increase it. The first non-trivial case is when $q = i - 1$. The next case is when $q = i$ and this is only non-trivial if $j = i + 1$. The next case is when $q = j - 1$ and $j \neq i + 1$. The remaining values are all trivial.

When $q = i-1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq i-1$.

$$\begin{aligned}\Phi_{\sigma_q}(x_{ij}\, t_k) &= p_{i-1,i}\, x_{i-1,j}\, \underline{p_{i-1,i}^{-1}\, t_{k'}} \overset{\text{(C-}pt\text{)}}{=} p_{i-1,i}\, \underline{x_{i-1,j}\, t_{k'}}\, p_{i-1,i}^{-1} \\ &\overset{\text{(C-}xt\text{)}}{=} \underline{p_{i-1,i}\, t_{k'}}\, x_{i-1,j}\, p_{i-1,i}^{-1} \overset{\text{(C-}pt\text{)}}{=} t_{k'}\, p_{i-1,i}\, x_{i-1,j}\, p_{i-1,i}^{-1} = \Phi_{\sigma_q}(t_k\, x_{ij})\end{aligned}$$

When $q = i$ and $j = i+1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq j$.

$$\begin{aligned}\Phi_{\sigma_q}(x_{ij}\, t_k) &= t_j^{-1}\, y_{ij}\, \underline{t_j\, t_{k'}} \overset{\text{(C-}tt\text{)}}{=} t_j^{-1}\, \underline{y_{ij}\, t_{k'}}\, t_j \\ &\overset{\text{(C-}yt\text{)}}{=} \underline{t_j^{-1}\, t_{k'}}\, y_{ij}\, t_j \overset{\text{(C-}tt\text{)}}{=} t_{k'}\, t_j^{-1}\, y_{ij}\, t_j = \Phi_{\sigma_q}(t_k\, x_{ij})\end{aligned}$$

When $q = j-1$ and $j \neq i+1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq i$.

$$\begin{aligned}\Phi_{\sigma_q}(x_{ij}\, t_k) &= p_{j-1,j}\, x_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, t_{k'}} \overset{\text{(C-}pt\text{)}}{=} p_{j-1,j}\, \underline{x_{i,j-1}\, t_{k'}}\, p_{j-1,j}^{-1} \\ &\overset{\text{(C-}xt\text{)}}{=} \underline{p_{j-1,j}\, t_{k'}}\, x_{i,j-1}\, p_{j-1,j}^{-1} \overset{\text{(C-}pt\text{)}}{=} t_{k'}\, p_{j-1,j}\, x_{i,j-1}\, p_{j-1,j}^{-1} = \Phi_{\sigma_q}(t_k\, x_{ij})\end{aligned}$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = i$.

$$\Phi_{\tau_q}(x_{ij}\, t_k) = x_{ij}^{-1}\, \underline{p_{ij}\, t_k} \overset{\text{(C-}pt\text{)}}{=} \underline{x_{ij}^{-1}\, t_k}\, p_{ij} \overset{\text{(C-}xt\text{)}}{=} t_k\, x_{ij}^{-1}\, p_{ij} = \Phi_{\tau_q}(t_k\, x_{ij})$$

Case 2. (C-yt) $\quad y_{ij}\, t_k = t_k\, y_{ij} \quad k \neq j,\ i < j$

First consider Φ_{σ_q}, the non-trivial cases are $q = i-1$, $q = i$ and $j = i+1$, and $q = j-1$ and $j \neq i+1$.

When $q = i-1$, we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$, where $k' \neq j$.

$$\begin{aligned}\Phi_{\sigma_q}(y_{ij}\, t_k) &= p_{i-1,i}\, y_{i-1,j}\, \underline{p_{i-1,i}^{-1}\, t_{k'}} \overset{\text{(C-}pt\text{)}}{=} p_{i-1,i}\, \underline{y_{i-1,j}\, t_{k'}}\, p_{i-1,i}^{-1} \\ &\overset{\text{(C-}yt\text{)}}{=} \underline{p_{i-1,i}\, t_{k'}}\, y_{i-1,j}\, p_{i-1,i}^{-1} \overset{\text{(C-}pt\text{)}}{=} t_{k'}\, p_{i-1,i}\, y_{i-1,j}\, p_{i-1,i}^{-1} = \Phi_{\sigma_q}(t_k\, y_{ij})\end{aligned}$$

When $q = i$ and $j = i+1$, we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq i$.

$$\Phi_{\sigma_q}(y_{ij}\, t_k) = \underline{x_{ij}\, t_{k'}} \overset{\text{(C-}xt\text{)}}{=} t_{k'}\, x_{ij} = \Phi_{\sigma_q}(t_k\, y_{ij})$$

When $q = j-1$ and $j \neq i+1$, we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq j-1$.

$$\begin{aligned}\Phi_{\sigma_q}(y_{ij}\, t_k) &= p_{j-1,j}\, y_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, t_{k'}} \overset{\text{(C-}pt\text{)}}{=} p_{j-1,j}\, \underline{y_{i,j-1}\, t_{k'}}\, p_{j-1,j}^{-1} \\ &\overset{\text{(C-}yt\text{)}}{=} \underline{p_{j-1,j}\, t_{k'}}\, y_{i,j-1}\, p_{j-1,j}^{-1} \overset{\text{(C-}pt\text{)}}{=} t_{k'}\, p_{j-1,j}\, y_{i,j-1}\, p_{j-1,j}^{-1} = \Phi_{\sigma_q}(t_k\, y_{ij})\end{aligned}$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = j$.

$$\Phi_{\tau_q}(y_{ij}\, t_k) = y_{ij}^{-1}\, \underline{p_{ij}\, t_k} \overset{\text{(C-}pt\text{)}}{=} \underline{y_{ij}^{-1}\, t_k}\, p_{ij} \overset{\text{(C-}yt\text{)}}{=} t_k\, y_{ij}^{-1}\, p_{ij} = \Phi_{\tau_q}(t_k\, y_{ij})$$

Case 3. (C1) $\quad \alpha_{ij}\, \beta_{kl} = \beta_{kl}\, \alpha_{ij} \quad (i, j, k, l)$ cyclically ordered

First consider Φ_{σ_q}. The non-trivial cases are $q = i-1$ and $i \neq l+1$, $q = i$ and $j = i+1$, $q = j-1$ and $j \neq i+1$, $q = j$ and $k = j+1$, $q = k-1$ and $j \neq k-1$, $q = k$ and $l = k+1$, $p = l-1$ and $l \neq k+1$, and $p = l$ and $i = l+1$.

When $q = i-1$ and $i \neq l+1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{kl}) = p_{i-1,i}\,\alpha_{i-1,j}\,\underline{p_{i-1,i}^{-1}\,\beta_{kl}} \overset{(C1)}{=} p_{i-1,i}\,\underline{\alpha_{i-1,j}\,\beta_{kl}}\,p_{i-1,i}^{-1}$$
$$\overset{(C1)}{=} \underline{p_{i-1,i}\,\beta_{kl}}\,\alpha_{i-1,j}\,p_{i-1,i}^{-1} \overset{(C1)}{=} \beta_{kl}\,p_{i-1,i}\,\alpha_{i-1,j}\,p_{i-1,i}^{-1} = \Phi_{\sigma_q}(\beta_{kl}\,\alpha_{ij})$$

When $q = i$ and $j = i+1$, the only non-trivial case is when $\alpha = x$.

$$\Phi_{\sigma_q}(x_{ij}\,\beta_{kl}) = t_j^{-1}\,y_{ij}\,\underline{t_j\,\beta_{kl}} \overset{(C\text{-}\beta t)}{=} t_j^{-1}\,\underline{y_{ij}\,\beta_{kl}}\,t_j$$
$$\overset{(C1)}{=} \underline{t_j^{-1}\,\beta_{kl}}\,y_{ij}\,t_j \overset{(C\text{-}\beta t)}{=} \beta_{kl}\,t_j^{-1}\,y_{ij}\,t_j = \Phi_{\sigma_q}(\beta_{kl}\,x_{ij})$$

When $q = j-1$ and $j \neq i+1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{kl}) = p_{j-1,j}\,\alpha_{i,j-1}\,\underline{p_{j-1,j}^{-1}\,\beta_{kl}} \overset{(C1)}{=} p_{j-1,j}\,\underline{\alpha_{i,j-1}\,\beta_{kl}}\,p_{j-1,j}^{-1}$$
$$\overset{(C1)}{=} \underline{p_{j-1,j}\,\beta_{kl}}\,\alpha_{i,j-1}\,p_{j-1,j}^{-1} \overset{(C1)}{=} \beta_{kl}\,p_{j-1,j}\,\alpha_{i,j-1}\,p_{j-1,j}^{-1} = \Phi_{\sigma_q}(\beta_{kl}\,\alpha_{ij})$$

When $q = j$ and $k = j+1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{kl}) = \underline{\alpha_{ik}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}} \overset{(C3)}{=} p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,\alpha_{ik} = \Phi_{\sigma_q}(\beta_{kl}\,\alpha_{ij})$$

When $q = k-1$ and $j \neq k-1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{kl}) = \underline{\alpha_{ij}\,p_{k-1,k}}\,\beta_{k-1,l}\,p_{k-1,k}^{-1} \overset{(C1)}{=} p_{k-1,k}\,\underline{\alpha_{ij}\,\beta_{k-1,l}}\,p_{k-1,k}^{-1}$$
$$\overset{(C1)}{=} p_{k-1,k}\,\beta_{k-1,l}\,\underline{\alpha_{ij}\,p_{k-1,k}^{-1}} \overset{(C1)}{=} p_{k-1,k}\,\beta_{k-1,l}\,p_{k-1,k}^{-1}\,\alpha_{ij} = \Phi_{\sigma_q}(\beta_{kl}\,\alpha_{ij})$$

When $q = k$ and $l = k+1$, the only non-trivial case is when $\beta = x$.

$$\Phi_{\sigma_q}(\alpha_{ij}\,x_{kl}) = \underline{\alpha_{ij}\,t_l^{-1}}\,y_{kl}\,t_l \overset{(C\text{-}\alpha t)}{=} t_l^{-1}\,\underline{\alpha_{ij}\,y_{kl}}\,t_l$$
$$\overset{(C1)}{=} t_l^{-1}\,y_{kl}\,\underline{\alpha_{ij}\,t_l} \overset{(C\text{-}\alpha t)}{=} t_l^{-1}\,y_{kl}\,t_l\,\alpha_{ij} = \Phi_{\sigma_q}(x_{kl}\,\alpha_{ij})$$

When $q = l-1$ and $l \neq k+1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{kl}) = \underline{\alpha_{ij}\,p_{l-1,l}}\,\beta_{k,l-1}\,p_{l-1,l}^{-1} \overset{(C1)}{=} p_{l-1,l}\,\underline{\alpha_{ij}\,\beta_{k,l-1}}\,p_{l-1,l}^{-1}$$
$$\overset{(C1)}{=} p_{l-1,l}\,\beta_{k,l-1}\,\underline{\alpha_{ij}\,p_{l-1,l}^{-1}} \overset{(C1)}{=} p_{l-1,l}\,\beta_{k,l-1}\,p_{l-1,l}^{-1}\,\alpha_{ij} = \Phi_{\sigma_q}(\beta_{kl}\,\alpha_{ij})$$

Finally, when $q = l$ and $i = l+1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{kl}) = \underline{p_{il}\,\alpha_{jl}\,p_{il}^{-1}\,\beta_{ik}} \overset{(C3)}{=} \beta_{ik}\,p_{il}\,\alpha_{jl}\,p_{il}^{-1} = \Phi_{\sigma_q}(\beta_{kl}\,\alpha_{ij})$$

Now consider Φ_{τ_q}, there are two non-trivial cases. In the first case $\Phi_{\tau_q}(\alpha_{ij}) = \alpha_{ij}^{-1}p_{ij}$ and we have the following.

$$\Phi_{\tau_q}(\alpha_{ij}\,\beta_{kl}) = \alpha_{ij}^{-1}\,\underline{p_{ij}\,\beta_{kl}} \overset{(C1)}{=} \underline{\alpha_{ij}^{-1}\,\beta_{kl}}\,p_{ij} \overset{(C1)}{=} \beta_{kl}\,\alpha_{ij}^{-1}\,p_{ij} = \Phi_{\tau_q}(\beta_{kl}\,\alpha_{ij})$$

In the second case $\Phi_{\tau_q}(\beta_{kl}) = \beta_{kl}^{-1}p_{kl}$ and we have the following.

$$\Phi_{\tau_q}(\alpha_{ij}\,\beta_{kl}) = \underline{\alpha_{ij}\,\beta_{kl}^{-1}}\,p_{kl} \overset{(C1)}{=} \beta_{kl}^{-1}\,\underline{\alpha_{ij}\,p_{kl}} \overset{(C1)}{=} \beta_{kl}^{-1}\,p_{kl}\,\alpha_{ij} = \Phi_{\tau_q}(\beta_{kl}\,\alpha_{ij})$$

Case 4. (C2) $\qquad \alpha_{ij}\ \beta_{ik}\,\gamma_{jk} = \beta_{ik}\,\gamma_{jk}\ \alpha_{ij} \qquad$ (i,j,k) cyclically ordered, (α,β,γ) as in Table 1

First consider Φ_{σ_q}. The only non-trivial cases are when $q = i-1$ and $i \neq k+1$, $q = i$ and $j = i+1$, $q = j-1$ and $j \neq i+1$, $q = j$ and $k = j+1$, $q = k-1$ and $k \neq j+1$, and $q = k$ and $i = k+1$.

When $q = i-1$ and $i \neq k+1$, we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= p_{i-1,i}\,\alpha_{i-1,j}\,\beta_{i-1,k}\,\underline{p^{-1}_{i-1,i}\,\gamma_{jk}} && \text{(C1)}\\
&= p_{i-1,i}\,\underline{\alpha_{i-1,j}\,\beta_{i-1,k}\,\gamma_{jk}}\,p^{-1}_{i-1,i} && \text{(C2)}\\
&= p_{i-1,i}\,\beta_{i-1,k}\,\underline{\gamma_{jk}}\,\alpha_{i-1,j}\,p^{-1}_{i-1,i} && \text{(C1)}\\
&= p_{i-1,i}\,\beta_{i-1,k}\,p^{-1}_{i-1,i}\,\gamma_{jk}\,p_{i-1,i}\,\alpha_{i-1,j}\,p^{-1}_{i-1,i} = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

When $q = i$ and $j = i+1$, we have two cases. Except for when $i < j < k$ and $(\alpha,\beta,\gamma) = (x,x,p)$ or $k < i < j$ and $(\alpha,\beta,\gamma) = (x,y,p)$, we have the following deduction. Let $\bar{t}_j$ and $\bar{\alpha}_{ij}$ be defined as follows.

$$
\bar{t}_j = \begin{cases} t_j & \text{if } \alpha = x \\ 1 & \text{if } \alpha \neq x \end{cases} \qquad \bar{\alpha}_{ij} = \begin{cases} p_{ij} & \text{if } \alpha = p \\ y_{ij} & \text{if } \alpha = x \\ x_{ij} & \text{if } \alpha = y \end{cases}
$$

So we have that $\Phi_{\sigma_q}(\alpha_{ij}) = \bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\bar{t}_j$.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\underline{\bar{t}_j\,\beta_{jk}\,p_{ij}\,\gamma_{ik}\,p_{ij}^{-1}}\\
& && \text{(C-}\beta t\text{) (C-}pt\text{) (C-}\gamma t\text{) (C-}pt\text{)}\\
&= \bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\underline{\beta_{jk}\,p_{ij}\,\gamma_{ik}\,p_{ij}^{-1}}\,\bar{t}_j && \text{(C2)}\\
&= \bar{t}_j^{-1}\,\underline{\bar{\alpha}_{ij}\,\gamma_{ik}\,\beta_{jk}}\,\bar{t}_j && \text{(C2)}\\
&= \underline{\bar{t}_j^{-1}\,\gamma_{ik}\,\beta_{jk}}\,\bar{\alpha}_{ij}\,\bar{t}_j && \text{(C-}pt\text{) (C-}pt\text{)}\\
&= \underline{\gamma_{ik}\,\beta_{jk}\,p_{ij}}\,p_{ij}^{-1}\,\bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\bar{t}_j && \text{(C2)}\\
&= \beta_{jk}p_{ij}\,\gamma_{ik}\,p_{ij}^{-1}\,\bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\bar{t}_j = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

When $i < j < k$ and $(\alpha,\beta,\gamma) = (x,x,p)$ or $k < i < j$ and $(\alpha,\beta,\gamma) = (x,y,p)$, we have the following deduction with $\beta = x$ or y respectively.

$$
\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\,\beta_{ik}\,p_{jk}) &= t_j^{-1}\,y_{ij}\,t_j\,\underline{\beta_{jk}\,p_{ij}\,p_{ik}}\,p_{ij}^{-1} && \text{(C2)}\\
&= \underline{t_j^{-1}\,y_{ij}\,t_jp_{ij}}\,p_{ik}\,\beta_{jk}\,p_{ij}^{-1} && \text{(C-}ypt\text{)}\\
&= p_{ij}\,\underline{y_{ij}\,p_{ik}\,\beta_{jk}}\,p_{ij}^{-1} && \text{(C2)}\\
&= \underline{p_{ij}\,p_{ik}\,\beta_{jk}}\,y_{ij}\,p_{ij}^{-1} && \text{(C2)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,y_{ij}\,p_{ij}^{-1} && \text{(C-}pt\text{)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,p_{ij}^{-1}\,t_j^{-1}\,\underline{p_{ij}\,t_j\,y_{ij}}\,p_{ij}^{-1} && \text{(C-}ypt\text{)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,p_{ij}^{-1}\,t_j^{-1}\,y_{ij}\,\underline{p_{ij}\,t_j}\,p_{ij}^{-1} && \text{(C-}pt\text{)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,p_{ij}^{-1}\,t_j^{-1}\,y_{ij}\,t_j = \Phi_{\sigma_q}(\beta_{ik}\,p_{jk}\,x_{ij})
\end{aligned}
$$

When $q = j-1$ and $j \neq i+1$, we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= p_{j-1,j}\,\alpha_{i,j-1}\,\underline{p_{j-1,j}^{-1}\,\beta_{ik}\,p_{j-1,j}}\,\gamma_{j-1,k}\,p_{j-1,j}^{-1} && \text{(C1)}\\
&= p_{j-1,j}\,\underline{\alpha_{i,j-1}\,\beta_{ik}\,\gamma_{j-1,k}}\,p_{j-1,j}^{-1} && \text{(C2)}\\
&= \underline{p_{j-1,j}\,\beta_{ik}}\,\gamma_{j-1,k}\,\alpha_{i,j-1}\,p_{j-1,j}^{-1} && \text{(C1)}\\
&= \beta_{ik}\,p_{j-1,j}\,\gamma_{j-1,k}\,\alpha_{i,j-1}\,p_{j-1,j}^{-1} = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}$$

When $q = j$ and $k = j+1$, we have two cases. Except for when $\gamma = x$, i.e. when $i < j < k$ and $(\alpha, \beta, \gamma) = (y, p, x)$ or $j < k < i$ and $(\alpha, \beta, \gamma) = (x, p, x)$, we have the following. Here

$$\bar{\gamma}_{jk} = \begin{cases} p_{jk} & \text{if } \gamma = p \\ x_{jk} & \text{if } \gamma = y \end{cases}$$

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \alpha_{ik}\,\underline{p_{jk}\beta_{ij}\,p_{jk}^{-1}}\,\bar{\gamma}_{jk} && \text{(C2)}\\
&= \alpha_{ik}\,p_{ik}^{-1}\,\underline{\beta_{ij}\,p_{ik}\,\bar{\gamma}_{jk}} && \text{(C2)}\\
&= \underline{\alpha_{ik}\,\bar{\gamma}_{jk}\,\beta_{ij}} && \text{(C2)}\\
&= \underline{\bar{\gamma}_{jk}\,\beta_{ij}}\,\alpha_{ik} && \text{(C2)}\\
&= \underline{p_{ik}^{-1}\,\beta_{ij}\,p_{ik}}\,\bar{\gamma}_{jk}\,\alpha_{ik} && \text{(C2)}\\
&= p_{jk}\,\beta_{ij}\,p_{jk}^{-1}\,\bar{\gamma}_{jk}\,\alpha_{ik} = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}$$

When $i < j < k$ and $(\alpha, \beta, \gamma) = (y, p, x)$ or when $j < k < i$ and $(\alpha, \beta, \gamma) = (x, p, x)$, we have

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,p_{ik}\,x_{jk}) &= \alpha_{ik}\,p_{jk}\,p_{ij}\,\underline{p_{jk}^{-1}\,t_k^{-1}\,y_{jk}\,t_k} && \text{(C-}ypt\text{)}\\
&= \underline{\alpha_{ik}\,p_{jk}\,p_{ij}}\,y_{jk}\,p_{jk}^{-1} && \text{(C2)}\\
&= p_{jk}\,p_{ij}\,\underline{\alpha_{ik}\,y_{jk}}\,p_{jk}^{-1} && \text{(C2)}\\
&= p_{jk}\,p_{ij}\,y_{jk}\,\underline{p_{ij}\,\alpha_{ik}\,p_{ij}^{-1}\,p_{jk}^{-1}} && \text{(C2)}\\
&= p_{jk}\,p_{ij}\,\underline{y_{jk}\,p_{jk}^{-1}}\,\alpha_{ik} && \text{(C-}ypt\text{)}\\
&= p_{jk}\,p_{ij}\,p_{jk}^{-1}\,t_k^{-1}\,y_{jk}\,t_k\,\alpha_{ik} = \Phi_{\sigma_q}(p_{ik}\,x_{jk}\,\alpha_{ij})
\end{aligned}$$

When $q = k-1$ and $k \neq j+1$, we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \underline{\alpha_{ij}\,p_{k-1,k}}\,\beta_{i,k-1}\,\gamma_{j,k-1}\,p_{k-1,k}^{-1} && \text{(C1)}\\
&= p_{k-1,k}\,\underline{\alpha_{ij}\,\beta_{i,k-1}\,\gamma_{j,k-1}}\,p_{k-1,k}^{-1} && \text{(C2)}\\
&= p_{k-1,k}\,\beta_{i,k-1}\,\gamma_{j,k-1}\,\underline{\alpha_{ij}\,p_{k-1,k}^{-1}} && \text{(C1)}\\
&= p_{k-1,k}\,\beta_{i,k-1}\,\gamma_{j,k-1}\,p_{k-1,k}^{-1}\,\alpha_{ij} = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}$$

Finally, when $q = k$ and $i = k + 1$, we have the following two cases. If $\beta \neq x$ then we have the following. Here

$$\bar{\beta}_{ik} = \begin{cases} p_{ik} & \text{if } \beta = p \\ x_{ik} & \text{if } \beta = y \end{cases}$$

$$\begin{aligned} \Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \underline{p_{ik}\,\alpha_{jk}\,p_{ik}^{-1}}\,\bar{\beta}_{ik}\,\gamma_{ij} \overset{(\text{C2})}{=} \underline{p_{ij}^{-1}\,\alpha_{jk}\,p_{ij}\,\bar{\beta}_{ik}}\,\gamma_{ij} \\ &\overset{(\text{C2})}{=} \bar{\beta}_{ik}\,\underline{\alpha_{jk}\,\gamma_{ij}} \overset{(\text{C2})}{=} \bar{\beta}_{ik}\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij}) \end{aligned}$$

And if $\beta = x$ then we have the following.

$$\begin{aligned} \Phi_{\sigma_q}(\alpha_{ij}\,x_{ik}\,\gamma_{jk}) &= p_{ik}\,\alpha_{jk}\,\underline{p_{ik}^{-1}\,t_i^{-1}}\,y_{ik}\,t_i\,\gamma_{ij} && (\text{C-}pt) \\ &= p_{ik}\,\alpha_{jk}\,\underline{t_i^{-1}\,p_{ik}^{-1}\,y_{ik}}\,t_i\,\gamma_{ij} && (\text{C-}ypt) \\ &= p_{ik}\,\alpha_{jk}\,y_{ik}\,\underline{t_i^{-1}\,p_{ik}^{-1}\,t_i}\,\gamma_{ij} && (\text{C-}pt) \\ &= p_{ik}\,\alpha_{jk}\,y_{ik}\,\underline{p_{ik}^{-1}\,\gamma_{ij}} && (\text{C2}) \\ &= p_{ik}\,\alpha_{jk}\,\underline{y_{ik}\,p_{jk}\,\gamma_{ij}\,p_{jk}^{-1}}\,p_{ik}^{-1} && (\text{C2}) \\ &= p_{ik}\,\underline{\alpha_{jk}\,\gamma_{ij}\,y_{ik}}\,p_{ik}^{-1} && (\text{C2}) \\ &= p_{ik}\,\underline{\gamma_{ij}\,y_{ik}}\,\alpha_{jk}\,p_{ik}^{-1} && (\text{C2}) \\ &= p_{ik}\,y_{ik}\,\underline{p_{jk}\,\gamma_{ij}\,p_{jk}^{-1}}\,\alpha_{jk}\,p_{ik}^{-1} && (\text{C2}) \\ &= p_{ik}\,y_{ik}\,\underline{p_{ik}^{-1}}\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} && (\text{C-}pt) \\ &= p_{ik}\,\underline{y_{ik}\,t_i^{-1}\,p_{ik}^{-1}}\,t_i\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} && (\text{C-}ypt) \\ &= \underline{p_{ik}\,t_i^{-1}\,p_{ik}^{-1}}\,y_{ik}\,t_i\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} && (\text{C-}pt) \\ &= t_i^{-1}\,y_{ik}\,t_i\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} = \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij}) \end{aligned}$$

Now consider Φ_{τ_q}, the non-trivial cases are as follows.

$q = i$	$i < j < k$	(x, p, p)	(x, y, y)	(x, x, p)
	$j < k < i$	(y, p, p)	(y, x, y)	(y, y, p)
	$k < i < j$	(x, p, p)	(x, x, x)	(x, y, p)
$q = j$	$i < j < k$	(y, p, p)	(y, y, y)	(y, p, x)
	$j < k < i$	(x, p, p)	(x, x, y)	(x, p, x)
	$k < i < j$	(y, p, p)	(y, x, x)	(y, p, y)
$q = k$	$i < j < k$	(p, y, y)	(x, y, y)	(y, y, y)
	$j < k < i$	(p, x, y)	(x, x, y)	(y, x, y)
	$k < i < j$	(p, x, x)	(x, x, x)	(y, x, x)

For the first two columns of the cases $q = i$ and $q = j$, we have the following.

$$\begin{aligned} \Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \alpha_{ij}^{-1}\,\underline{p_{ij}\,\beta_{ik}\,\gamma_{jk}} \overset{(\text{C2})}{=} \underline{\alpha_{ij}^{-1}\,\beta_{ik}\,\gamma_{jk}}\,p_{ij} \\ &\overset{(\text{C2})}{=} \beta_{ik}\,\gamma_{jk}\,\alpha_{ij}^{-1}\,p_{ij} = \Phi_{\tau_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij}) \end{aligned}$$

For the third column in the case $q = i$, we have the following.

$$\Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,p_{jk}) = \alpha_{ij}^{-1}\,p_{ij}\,\beta_{ik}^{-1}\,\underline{p_{ik}\,p_{jk}} \overset{\text{(C2)}}{=} \alpha_{ij}^{-1}\,\underline{p_{ij}\,\beta_{ik}^{-1}\,p_{ij}^{-1}}\,p_{ik}\,p_{jk}\,p_{ij}$$
$$\overset{\text{(C2)}}{=} \underline{\alpha_{ij}^{-1}\,p_{jk}^{-1}\,\beta_{ik}^{-1}\,p_{jk}}\,p_{ik}\,p_{jk}\,p_{ij} \overset{\text{(C2)}}{=} \beta_{ik}^{-1}\,\underline{\alpha_{ij}^{-1}\,p_{ik}}\,p_{jk}\,p_{ij}$$
$$\overset{\text{(C2)}}{=} \beta_{ik}^{-1}\,p_{ik}\,p_{jk}\,\alpha_{ij}^{-1}\,p_{ij} = \Phi_{\tau_q}(\beta_{ik}\,y_{jk}\,\alpha_{ij})$$

For the third column in the case $q = j$, we have the following.

$$\Phi_{\tau_q}(\alpha_{ij}\,p_{ik}\,\gamma_{jk}) = \alpha_{ij}^{-1}\,\underline{p_{ij}\,p_{ik}\,\gamma_{jk}^{-1}}\,p_{jk} \overset{\text{(C2)}}{=} \alpha_{ij}^{-1}\,\gamma_{jk}^{-1}\,\underline{p_{ij}\,p_{ik}\,p_{jk}}$$
$$\overset{\text{(C2)}}{=} \underline{\alpha_{ij}^{-1}\,\gamma_{jk}^{-1}}\,p_{ik}\,p_{jk}\,p_{ij} \overset{\text{(C2)}}{=} p_{ik}\,\gamma_{jk}^{-1}\,\underline{p_{ik}^{-1}\,\alpha_{ij}^{-1}\,p_{ik}\,p_{jk}}\,p_{ij}$$
$$\overset{\text{(C2)}}{=} p_{ik}\,\gamma_{jk}^{-1}\,p_{jk}\,\alpha_{ij}^{-1}\,p_{ij} = \Phi_{\tau_q}(p_{ik}\,\gamma_{jk}\,\alpha_{ij})$$

For the case when $q = k$, we have the following.

$$\Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) = \alpha_{ij}\,\beta_{ik}^{-1}\,\underline{p_{ik}\,\gamma_{jk}^{-1}}\,p_{jk} \overset{\text{(C2)}}{=} \alpha_{ij}\,\underline{\beta_{ik}^{-1}\,p_{ij}^{-1}\,\gamma_{jk}^{-1}\,p_{ij}}\,p_{ik}\,p_{jk}$$
$$\overset{\text{(C2)}}{=} \underline{\alpha_{ij}\,\gamma_{jk}^{-1}\,\beta_{ik}^{-1}}\,p_{ik}\,p_{jk} \overset{\text{(C2)}}{=} \gamma_{jk}^{-1}\,\beta_{ik}^{-1}\,\underline{\alpha_{ij}\,p_{ik}\,p_{jk}} \overset{\text{(C2)}}{=} \underline{\gamma_{jk}^{-1}\,\beta_{ik}^{-1}}\,p_{ik}\,p_{jk}\,\alpha_{ij}$$
$$\overset{\text{(C2)}}{=} \beta_{ik}^{-1}\,\underline{p_{ij}^{-1}\,\gamma_{jk}^{-1}\,p_{ij}\,p_{ik}}\,p_{jk}\,\alpha_{ij} \overset{\text{(C2)}}{=} \beta_{ik}^{-1}\,p_{ik}\,\gamma_{jk}^{-1}\,p_{jk}\,\alpha_{ij} = \Phi_{\tau_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})$$

Case 5. (C3) $\quad \alpha_{ik}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1} = p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,\alpha_{ik} \quad$ (i, j, k, l) cyclically ordered

First consider Φ_{σ_q}. As before the only non-trivial cases are when $q = i - 1$ and $i \neq l + 1$, $q = i$ and $j = i + 1$, $q = j - 1$ and $j \neq i + 1$, $q = j$ and $k = j + 1$, $q = k - 1$ and $k \neq j + 1$, $q = k$ and $l = k + 1$, $q = l - 1$ and $l \neq k + 1$, and $q = l$ and $i = l + 1$.

When $q = i - 1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}) = p_{i-1,i}\,\alpha_{i-1,k}\,\underline{p_{i-1,i}^{-1}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}}$$
$$\overset{\text{(C1)(C1)(C1)}}{=} p_{i-1,i}\,\underline{\alpha_{i-1,k}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}}\,p_{i-1,i}^{-1} \overset{\text{(C3)}}{=} \underline{p_{i-1,i}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}}\,\alpha_{i-1,k}\,p_{i-1,i}^{-1}$$
$$\overset{\text{(C1)(C1)(C1)}}{=} p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,p_{i-1,i}\,\alpha_{i-1,k}\,p_{i-1,i}^{-1} = \Phi_{\sigma_q}(p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,\alpha_{ik})$$

When $q = i$ and $j = i + 1$, we have the following. (Here the (C2)s hold because we are in either of the bottom two rows of Table 1, both of which contain (α, p, p) for $\alpha = p$, x, and y.)

$$\Phi_{\sigma_q}(\alpha_{ik}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}) = \underline{\alpha_{jk}\,p_{ij}\,p_{ik}}\,\beta_{il}\,p_{ik}^{-1}\,p_{ij}^{-1} \overset{\text{(C2)}}{=} p_{ij}\,p_{ik}\,\underline{\alpha_{jk}\,\beta_{il}}\,p_{ik}^{-1}\,p_{ij}^{-1}$$
$$\overset{\text{(C1)}}{=} p_{ij}\,p_{ik}\,\beta_{il}\,\underline{\alpha_{jk}\,p_{ik}^{-1}\,p_{ij}^{-1}} \overset{\text{(C2)}}{=} p_{ij}\,p_{ik}\,\beta_{il}\,p_{ik}^{-1}\,p_{ij}^{-1}\,\alpha_{jk} = \Phi_{\sigma_q}(p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,\alpha_{ik})$$

When $q = j-1$ and $j \neq i+1$, we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{\alpha_{ik}\, p_{j-1,j}}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}\, p_{j-1,j}^{-1} \\
&\overset{(C1)}{=} p_{j-1,j}\, \underline{\alpha_{ik}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}}\, p_{j-1,j}^{-1} \overset{(C3)}{=} p_{j-1,j}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}\, \underline{\alpha_{ik}\, p_{j-1,j}^{-1}} \\
&\overset{(C1)}{=} p_{j-1,j}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}\, p_{j-1,j}^{-1}\, \alpha_{ik} = \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}$$

When $q = j$ and $k = j+1$, we have the following.

$$\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) = p_{jk}\, \underline{\alpha_{ij}\, \beta_{kl}}\, p_{jk}^{-1} \overset{(C1)}{=} p_{jk}\, \beta_{kl}\, \alpha_{ij}\, p_{jk}^{-1} = \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})$$

When $q = k-1$ and $k \neq j+1$, we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= p_{k-1,k}\, \alpha_{i,k-1}\, p_{j,k-1}\, \underline{p_{k-1,k}^{-1}\, \beta_{jl}}\, p_{k-1,k}\, p_{j,k-1}^{-1}\, p_{k-1,k}^{-1} \\
&\overset{(C1)}{=} p_{k-1,k}\, \underline{\alpha_{i,k-1}\, p_{j,k-1}\, \beta_{jl}\, p_{j,k-1}^{-1}}\, p_{k-1,k}^{-1} \overset{(C3)}{=} p_{k-1,k}\, p_{j,k-1}\, \underline{\beta_{jl}}\, p_{j,k-1}^{-1}\, \alpha_{i,k-1}\, p_{k-1,k}^{-1} \\
&\overset{(C1)}{=} p_{k-1,k}\, p_{j,k-1}\, p_{k-1,k}^{-1}\, \beta_{jl}\, p_{k-1,k}\, p_{j,k-1}^{-1}\, \alpha_{i,k-1}\, p_{k-1,k}^{-1} = \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}$$

When $q = k$ and $l = k+1$, we have the following. (Here the (C2)s hold because we are in either of the top two rows of Table 1, both of which contain (β, p, p) for $\beta = p$, x, and y.)

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \alpha_{il}\, \underline{p_{jl}\, p_{kl}\, \beta_{jk}\, p_{kl}^{-1}\, p_{jl}^{-1}} \overset{(C2)}{=} \underline{\alpha_{il}\, \beta_{jk}} \\
&\overset{(C1)}{=} \underline{\beta_{jk}}\, \alpha_{il} \overset{(C2)}{=} p_{jl}\, p_{kl}\, \beta_{jk}\, p_{kl}^{-1}\, p_{jl}^{-1}\, \alpha_{il} = \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}$$

When $q = l-1$ and $l \neq k+1$, we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{\alpha_{ik}\, p_{jk}\, p_{l,l-1}}\, \beta_{j,l-1}\, \underline{p_{l,l-1}^{-1}\, p_{jk}^{-1}} \\
&\overset{(C1)(C1)(C1)}{=} p_{l,l-1}\, \underline{\alpha_{ik}\, p_{jk}\, \beta_{j,l-1}\, p_{jk}^{-1}}\, p_{l,l-1}^{-1} \overset{(C3)}{=} \underline{p_{l,l-1}\, p_{jk}}\, \beta_{j,l-1}\, \underline{p_{jk}^{-1}\, \alpha_{ik}\, p_{l,l-1}^{-1}} \\
&\overset{(C1)(C1)(C1)}{=} p_{jk}\, p_{l,l-1}\, \beta_{j,l-1}\, p_{l,l-1}^{-1}\, p_{jk}^{-1}\, \alpha_{ik} = \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}$$

Finally, when $q = l$ and $i = l+1$, we have the following. (Here the (C2)s hold because they always hold for the triples (α, p, p) and (β, p, p).)

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{p_{il}\, \alpha_{kl}\, p_{il}^{-1}}\, \underline{p_{kj}\, \beta_{ij}\, p_{jk}^{-1}} \\
&\overset{(C2)(C2)}{=} p_{ik}^{-1}\, \underline{\alpha_{kl}\, \beta_{ij}}\, p_{ik} \overset{(C1)}{=} \underline{p_{ik}^{-1}\, \beta_{ij}\, p_{ik}}\, \underline{p_{ik}^{-1}\, \alpha_{kl}\, p_{ik}} \\
&\overset{(C2)(C2)}{=} p_{jk}\, \beta_{ij}\, p_{jk}^{-1}\, p_{kl}\, \alpha_{kl}\, p_{kl}^{-1} = \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}$$

Now consider Φ_{τ_q}, there are two non-trivial cases. In the first case $\Phi_{\tau_q}(\alpha_{ik}) = \alpha_{ik}^{-1} p_{ik}$ and we have the following.

$$\begin{aligned}
\Phi_{\tau_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \alpha_{ik}^{-1}\, \underline{p_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}} \overset{(C3)}{=} \underline{\alpha_{ik}^{-1}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}}\, p_{ik} \\
&\overset{(C3)}{=} p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik}^{-1}\, p_{ik} = \Phi_{\tau_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}$$

In the second case $\Phi_{\tau_q}(\beta_{jl}) = \beta_{jl}^{-1} p_{jl}$ and we have the following.

$$\Phi_{\tau_q}(\alpha_{ik}\ p_{jk}\,\beta_{jl}\,p_{jk}^{-1}) = \underline{\alpha_{ik}\,p_{jk}\,\beta_{jl}^{-1}}\,p_{jl}\,p_{jk}^{-1} \overset{(C3)}{=} p_{jk}\,\beta_{jl}^{-1}\,\underline{p_{jk}^{-1}\,\alpha_{ik}\,p_{jk}\,p_{jl}\,p_{jk}^{-1}} \overset{(C3)}{=} p_{jk}\,\beta_{jl}^{-1}\,p_{jl}\,p_{jk}^{-1}\,\alpha_{ik} = \Phi_{\tau_q}(p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\ \alpha_{ik})$$

Case 6. (C-*xpt*) $\qquad x_{ij}\,p_{ij}\,t_i = p_{ij}\,t_i\,x_{ij} \qquad i < j$

First consider Φ_{σ_q}. The only non-trivial cases are when $q = i-1$, $q = i$ and $j = i+1$, and $q = j-1$ and $j \neq i+1$.

When $q = i-1$, we have the following.

$$\begin{aligned}\Phi_{\sigma_q}(x_{ij}\,p_{ij}\,t_i) &= p_{i-1,i}\,x_{i-1,j}\,p_{i-1,j}\,\underline{p_{i-1,i}^{-1}\,t_{i-1}} \\ &\overset{(C\text{-}pt)}{=} p_{i-1,i}\,\underline{x_{i-1,j}\,p_{i-1,j}\,t_{i-1}}\,p_{i-1,i}^{-1} \overset{(C\text{-}xpt)}{=} p_{i-1,i}\,p_{i-1,j}\,\underline{t_{i-1}}\,x_{i-1,j}\,p_{i-1,i}^{-1} \\ &\overset{(C\text{-}pt)}{=} p_{i-1,i}\,p_{i-1,j}\,p_{i-1,i}^{-1}\,t_{i-1}\,p_{i-1,i}\,x_{i-1,j}\,p_{i-1,i}^{-1} = \Phi_{\sigma_q}(p_{ij}\,t_i\,x_{ij})\end{aligned}$$

When $q = i$ and $j = i+1$, we have the following.

$$\begin{aligned}\Phi_{\sigma_q}(x_{ij}\,p_{ij}\,t_i) &= t_j^{-1}\,y_{ij}\,\underline{t_j\,p_{ij}}\,t_j \overset{(C\text{-}pt)}{=} t_j^{-1}\,\underline{y_{ij}\,p_{ij}\,t_j}\,t_j \\ &\overset{(C\text{-}ypt)}{=} \underline{t_j^{-1}\,p_{ij}\,t_j}\,y_{ij}\,t_j \overset{(C\text{-}pt)}{=} p_{ij}\,y_{ij}\,t_j = \Phi_{\sigma_q}(p_{ij}\,t_i\,x_{ij})\end{aligned}$$

When $q = j-1$ and $j \neq i+1$, we have the following.

$$\begin{aligned}\Phi_{\sigma_q}(x_{ij}\,p_{ij}\,t_i) &= p_{j-1,j}\,x_{i,j-1}\,p_{i,j-1}\,\underline{p_{j-1,j}^{-1}\,t_i} \\ &\overset{(C\text{-}pt)}{=} p_{j-1,j}\,\underline{x_{i,j-1}\,p_{i,j-1}\,t_i}\,p_{j-1,j}^{-1} \overset{(C\text{-}xpt)}{=} p_{j-1,j}\,p_{i,j-1}\,\underline{t_i}\,x_{i,j-1}\,p_{j-1,j}^{-1} \\ &\overset{(C\text{-}pt)}{=} p_{j-1,j}\,p_{i,j-1}\,p_{j-1,j}^{-1}\,t_i\,p_{j-1,j}\,x_{i,j-1}\,p_{j-1,j}^{-1} = \Phi_{\sigma_q}(p_{ij}\,t_i\,x_{ij})\end{aligned}$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = i$.

$$\Phi_{\tau_q}(x_{ij}\,p_{ij}\,t_i) = x_{ij}^{-1} p_{ij}\,\underline{p_{ij}\,t_i} \overset{(C\text{-}pt)}{=} \underline{x_{ij}^{-1} p_{ij}\,t_i}\,p_{ij} \overset{(C\text{-}xpt)}{=} p_{ij}\,t_i\,x_{ij}^{-1} p_{ij} = \Phi_{\tau_q}(p_{ij}\,t_i\,x_{ij})$$

Case 7. (C-*ypt*) $\qquad y_{ij}\,p_{ij}\,t_j = p_{ij}\,t_j\,y_{ij} \qquad i < j$

First consider Φ_{σ_q}. The only non-trivial cases are when $q = i-1$, $q = i$ and $j = i+1$, and $q = j-1$ and $j \neq i+1$.

When $q = i-1$, we have the following.

$$\begin{aligned}\Phi_{\sigma_q}(y_{ij}\,p_{ij}\,t_j) &= p_{i-1,i}\,y_{i-1,j}\,p_{i-1,j}\,\underline{p_{i-1,i}^{-1}\,t_j} \\ &\overset{(C\text{-}pt)}{=} p_{i-1,i}\,\underline{y_{i-1,j}\,p_{i-1,j}\,t_j}\,p_{i-1,i}^{-1} \overset{(C\text{-}ypt)}{=} p_{i-1,i}\,p_{i-1,j}\,\underline{t_j}\,y_{i-1,j}\,p_{i-1,i}^{-1} \\ &\overset{(C\text{-}pt)}{=} p_{i-1,i}\,p_{i-1,j}\,p_{i-1,i}^{-1}\,t_j\,p_{i-1,i}\,y_{i-1,j}\,p_{i-1,i}^{-1} = \Phi_{\sigma_q}(p_{ij}\,t_j\,y_{ij})\end{aligned}$$

When $q = i$ and $j = i+1$, we have the following.

$$\Phi_{\sigma_q}(y_{ij}\,p_{ij}\,t_j) = \underline{x_{ij}\,p_{ij}\,t_i} \overset{(C\text{-}xpt)}{=} p_{ij}\,t_i\,x_{ij} = \Phi_{\sigma_q}(p_{ij}\,t_j\,y_{ij})$$

When $q = j-1$ and $j \neq i+1$, we have the following.

$$\Phi_{\sigma_q}(y_{ij}\, p_{ij}\, t_j) = p_{j-1,j}\, y_{i,j-1}\, p_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, t_{j-1}}$$
$$\overset{(\text{C-}pt)}{=} p_{j-1,j}\, \underline{y_{i,j-1}\, p_{i,j-1}\, t_{j-1}}\, p_{j-1,j}^{-1} \overset{(\text{C-}ypt)}{=} p_{j-1,j}\, p_{i,j-1}\, \underline{t_{j-1}}\, y_{i,j-1}\, p_{j-1,j}^{-1}$$
$$\overset{(\text{C-}pt)}{=} p_{j-1,j}\, p_{i,j-1}\, p_{j-1,j}^{-1}\, t_{j-1}\, p_{j-1,j}\, y_{i,j-1}\, p_{j-1,j}^{-1} = \Phi_{\sigma_q}(p_{ij}\, t_j\, y_{ij})$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = j$.

$$\Phi_{\tau_q}(y_{ij}\, p_{ij}\, t_j) = y_{ij}^{-1}\, p_{ij}\, \underline{p_{ij}\, t_j} \overset{(\text{C-}pt)}{=} \underline{y_{ij}^{-1}\, p_{ij}\, t_j}\, p_{ij} \overset{(\text{C-}ypt)}{=} p_{ij}\, t_j\, y_{ij}^{-1}\, p_{ij} = \Phi_{\sigma_q}(p_{ij}\, t_j\, y_{ij})$$

□

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References

[1] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24**(3–4) (1997), 235–265; Computational algebra and number theory (London, 1993).
[2] T. Brendle and A. Hatcher, *Configuration spaces of rings and wickets*, Comment. Math. Helv. **88**(1) (2013), 131–162.
[3] A. Hatcher and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology **19**(3) (1980), 221–237.
[4] H.M. Hilden, *Generators for two groups related to the braid group*, Pacific J. Math. **59**(2) (1975), 475–486.
[5] D. Margalit and J. McCammond, *Geometric presentations for the pure braid group*, J. Knot Theory Ramifications, **18**(1) (2009), 1–20.
[6] S. Tawn, *A presentation for Hilden's subgroup of the braid group*, Math. Res. Lett. **15**(6) (2008), 1277–1293.
[7] ———, *Erratum: a presentation for Hilden's subgroup of the braid group*, Math. Res. Lett. **18**(1) (2011), 175–180.
[8] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45**(2–3) (1983), 157–174.
[9] ———, *Mapping class group of a handlebody*, Fund. Math. **158**(3) (1998), 195–228.
[10] ———, *An elementary approach to the mapping class group of a surface*, Geom. Topol. **3** (1999), 405–466 (electronic).

University of Western Sydney, Centre for Research in Mathematics, Locked Bag 1797, Penrith NSW 2751, Australia
E-mail address: stephen@tawn.co.uk
URL: www.stephentawn.info