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A PRESENTATION FOR HILDEN'S SUBGROUP OF THE BRAID GROUP

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Abstract. Consider the unit ball, $B = D\times[0,1]$, containing n unknotted arcs $a_1,\dots,a_n$ such that the boundary of each a_i lies in $D\times\{0\}$. We give a finite presentation for the mapping class group of B fixing the arcs $\{a_1,\dots,a_n\}$ setwise and fixing $D\times\{1\}$ pointwise. This presentation is calculated using the action of this group on a simply-connected complex.

1. Introduction

Let H^3 denote the closed upper half-space of $\mathbb{R}^3$, let $a_1, a_2, \dots, a_n \subset H^3$ be n pairwise disjoint properly embedded unknotted arcs and let $a_* = a_1 \cup a_2 \cup \cdots \cup a_n$. Viewing the braid group as the mapping class group of the punctured disc, if this disc is included in ∂H^3 with ∂a_* as the punctures, one can define Hilden's group, $\mathbf{H}_{2n}$, to be the subgroup of $\mathbf{B}_{2n}$ consisting of all mapping classes that can be extended to $H^3 \setminus a_*$. Or equivalently, $\mathbf{H}_{2n}$ is the stabiliser of a_* under the action of $\mathbf{B}_{2n}$ on $0, 2n$–tangles.

Hilden[5] found generators for a similar subgroup of the braid group of a sphere. For any given braid b multiplying on either the left or the right by elements of $\mathbf{H}_{2n}$ preserves the plat closure, ie plat closure is constant on each double coset. Birman[1] showed that if two braids have the same plat closure then they can be related by a sequence of these double coset moves and stabilisation moves that changes the braid index by 2.

We calculate a presentation for $\mathbf{H}_{2n}$ using the action of this group on a cellular complex. Hatcher–Thurston[4], Wajnryb[7, 8, 9], Laudenbach[6], etc used the same method to calculate presentations for mapping class groups. We start in Section 2 by outlining this method. A similar but more general method is given by Brown [3]. Brendle–Hatcher[2] have calculated a presentation for $\mathbf{H}_{2n}$ using a different method.

In Section 3 we define a simply-connected complex $\overline{\mathbf{X}}_n$. In Section 4 we remove some of the edges and faces of this complex resulting in a new complex which remains simply-connected but gives a simpler presentation. This presentation is calculated in Section 5 and then used to calculate a presentation with generators similar to those found by Hilden.

2. The method

Suppose that X is a connected simply-connected cellular 2-complex such that each attaching map is injective and that each cell is uniquely determined by its boundary. Suppose that G is a group acting cellularly on the right of X, and that this action

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is transitive on the vertex set X^0. Pick a vertex $v_0 \in X^0$ as a basepoint and let H denote its stabiliser in G, ie $H = \{g \in G \mid v_0 \cdot g = v_0\}$. Suppose that H has a presentation with generating set S_0 and relations R_0, ie $H = \langle S_0 | R_0 \rangle$.

Given vertices $u, v \in X^0$ such that $\{u, v\}$ is the boundary of an edge of X we will write (u, v) for this (oriented) edge. Given a sequence $v_1, v_2, \ldots, v_k$ of vertices such that either $v_i = v_{i+1}$ or (v_i, v_{i+1}) forms an edge we will write $(v_1, v_2, \ldots, v_k)$ for the path traversing these edges. Whenever $v_i = v_{i+1}$ we shall say that v_i is a stationary point.

Let E denote the set of all oriented edges starting at v_0, so H acts on E. Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a set of representatives for the H–orbits of the edges in E, ie $E = \bigcup_{\lambda\in\Lambda} He_\lambda$ and $He_\lambda = He_{\lambda'}$ only if $\lambda = \lambda'$. Since the action of G is transitive on X^0 we can find $r_\lambda \in G$ such that $e_\lambda = (v_0, v_0 \cdot r_\lambda)$. Let $S_1 = \{r_\lambda\}_{\lambda\in\Lambda}$.

The edges $\{e_\lambda\}_{\lambda\in\Lambda}$ also form a set of representatives for the edge orbits of the G–action on X. To see this suppose that two of these edges lie in the same G–orbit, ie $(v_0, v) = (v_0, u) \cdot g$. Then we have that $v_0 = v_0 \cdot g$ therefore $g \in H$.

Suppose that $\{f_\mu\}_{\mu\in M}$ is a set of representatives for the G–orbits of the faces of X. Since the action is transitive on X^0, we may assume that the boundary of each face f_μ contains the vertex v_0.

Definition 2.1. An *h-product of length k* is a word of the form

$$h_{k+1}\ r_{\lambda_k} h_k\ r_{\lambda_{k-1}} h_{k-1}\ \cdots\ r_{\lambda_1} h_1$$

where each $\lambda_i \in \Lambda$ and each of the h_i are words in H. To each h-product we can associate an edge path $P = (v_0, v_1, \ldots, v_k)$ in X starting at v_0 then visiting the vertices $v_1 = v_0 \cdot r_{\lambda_1} h_1$, $v_2 = v_0 \cdot r_{\lambda_2} h_2\ r_{\lambda_1} h_1$, etc. This means that the edge (v_{i-1}, v_i) is in the orbit of $(v_0, v_0 \cdot r_{\lambda_i})$. Given any edge path starting at v_0 we can choose an h-product to represent it.

We can now choose the following three sets of relations.

- R_1: For each edge orbit representative e_λ pick a generating set T for the stabiliser of this edge, ie $\langle T \rangle = \mathrm{Stab}_G(v_0) \cap \mathrm{Stab}_G(v_0 \cdot r_\lambda)$. For each $t \in T$ we have the relation $r_\lambda t r_\lambda^{-1} = h$ for some word $h \in H$.
- R_2: For each e_λ we have a relation $r_{\lambda'} h\ r_\lambda = h'$ where the LHS is a choice of h-product for the path $(v_0, v_0 \cdot r_\lambda, v_0)$ and h' is some word in H.
- R_3: For each face orbit representative f_μ with boundary $(v_0, v_1, \ldots, v_{k-1}, v_0)$ choose an h-product representing this path and a word $h \in H$ such that $r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1 = h$.

Theorem 2.2. *The group G has the following presentation.*

$$G = \langle S_0 \cup S_1 | R_0 \cup R_1 \cup R_2 \cup R_3 \rangle$$

Corollary 2.3. *Suppose that H is finitely presented, that the number of edge and face orbits is finite and that each edge stabiliser is finitely generated. Then G has a finite presentation.*

We prove Theorem 2.2 in several steps.

Claim 1. *The set $S_0 \cup S_1$ generates G.*

Proof. Given any $g \in G$, let $v = v_0 \cdot g$. Now as X is connected there is an edge path connecting v_0 to v. Choose an h-product $g_1 = h_{k+1}\ r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1$ representing this path. Then $v_0 \cdot g g_1^{-1} = v_0$ so $g = h g_1$ for some $h \in H$. □

Claim 2. *If two h-products, p_1 and p_2, give rise to the same path and are equal in G then they are equivalent modulo $R_0 \cup R_1$.*

Proof. Because p_1 and p_2 represent the same path they must have equal length. Suppose that $p_1 = h_{k+1}\ r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1$ and $p_2 = f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_1} f_1$. Clearly, if the two h-products are of length 0 then they are both words in H and so are equivalent modulo R_0. Now suppose that $k \neq 0$. The fact that p_1 and p_2 represent the same path means that

$$(v_0,\ v_0 \cdot r_{\lambda_1} h_1,\ v_0 \cdot r_{\lambda_2} h_2\ r_{\lambda_1} h_1,\ \ldots) = (v_0,\ v_0 \cdot r_{\lambda'_1} f_1,\ v_0 \cdot r_{\lambda'_2} f_2\ r_{\lambda'_1} f_1,\ \ldots),$$

therefore

$$(v_0, v_0 \cdot r_{\lambda_1}) = (v_0, v_0 \cdot r_{\lambda'_1}) \cdot f_1 h_1^{-1}.$$

So $\lambda_1 = \lambda'_1$ and $f_1 h_1^{-1}$ is in the stabiliser of the edge e_{λ_1}. Hence, for some word f'_2 in H

$$f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_2} f_2\ r_{\lambda'_1} f_1 h_1^{-1} h_1 \quad = \quad f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_2} f'_2\ r_{\lambda_1} h_1$$

modulo R_1. By induction the two shorter h-products $h_{k+1}\ r_{\lambda_k} h_k \cdots r_{\lambda_2} h_2$ and $f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_2} f'_2$ are equivalent modulo $R_0 \cup R_1$, and so $p_1 = p_2$ modulo $R_0 \cup R_1$. □

Claim 3. *Suppose that two h-products represent the same element of G and induce edge paths that are equivalent modulo backtracking. Then they are equivalent modulo $R_0 \cup R_1 \cup R_2$.*

Proof. It is enough to show that any h-product is equivalent to an h-product that represents a path without any backtracking. Furthermore, if we proceed by induction on the length of the h-product, it is enough to show that any h-product whose associated path has backtracking at the end is equivalent to a shorter h-product.

Suppose that $g = h_{k+3}\ r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}} h_{k+1}\ g_k$ is such an h-product, ie

$$\begin{aligned} v_k &= v_0 \cdot g_k \\ v_{k+1} &= v_0 \cdot r_{\lambda_{k+1}} h_{k+1}\ g_k \\ v_{k+2} = v_k &= v_0 \cdot r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}} h_{k+1}\ g_k \end{aligned}$$

and g_k is a shorter h-product. So, multiplying by $g_k^{-1} h_{k+1}^{-1}$, we find that $r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}}$ is an h-product with associated path $(v_0, v_0 \cdot r_{\lambda_{k+1}}, v_0)$. Suppose that $r_{\lambda'} h r_\lambda = h'$ is the R_2 relation corresponding to this path. Then $\lambda = \lambda_{k+1}$ and $v_0 \cdot r_{\lambda'} h = v_0 \cdot r_{\lambda_{k+2}} h_{k+2}$. So $\lambda' = \lambda_{k+2}$ and $h_{k+2} h^{-1}$ is in the stabiliser of the edge $e_{\lambda_{k+1}}$. Therefore there exists a word f in H such that

$$h_{k+3}\ r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}} h_{k+1}\ g_k = h_{k+3} f\ r_{\lambda'} h\ r_\lambda h_{k+1}\ g_k$$

modulo R_1. Hence modulo R_2 this is equal to $h_{k+3} f h' h_{k+1} g_k$, a shorter h-product. □

Claim 4. *Any h-product equal to the identity in G is equivalent to the identity modulo $R_0 \cup R_1 \cup R_2 \cup R_3$.*

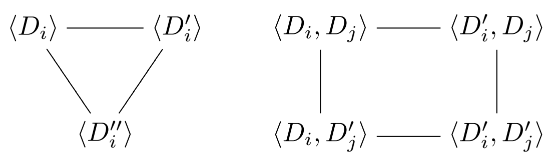
Proof. Given any h-product g_k equal to the identity in G its associated edge path must be a loop. Since X is simply-connected this loop is the boundary of a union of faces of X. So choose one of these faces f touching the loop at a vertex v then modulo $R_0 \cup R_1 \cup R_2$ we can add backtracking starting at v going around the boundary of f. Modulo R_3 we can remove one pass round ∂f. This leaves a new loop that can be spanned by one less face, which, by induction on the minimum number of faces needed to span a loop, is equivalent to the identity. □

Proof of Theorem 2.2. Given any word in the generators, $S_0 \cup S_1$, that is equal to the identity in G then modulo R_2 it is equivalent to an h-product and so by Claim 4 is equivalent to the identity modulo $R_0 \cup R_1 \cup R_2 \cup R_3$. □

3. The complex $\overline{\mathbf{X}}_n$

An embedded disc $D \subseteq H^3$ is said to *cut out* a_i if the interior of D is disjoint from a_*, the arc a_i is contained in the boundary of D and the boundary of D lies in $a_i \cup \partial H^3$, ie $a_i \subset \partial D$ and $\partial D \subset a_i \cup \partial H^3$. A *cut system for* a_* is the isotopy class of n pairwise disjoint discs $\langle D_1, D_2, \dots D_n\rangle$ where each D_i cuts out the arc a_i. Say that two cut systems $\langle D_1, D_2, \dots, D_n\rangle$ and $\langle E_1, E_2, \dots, E_n\rangle$ differ by a simple i-move if $D_i \cap E_i = a_i$ and $D_j = E_j$ for all $j \neq i$. If this is the case we will suppress the non-changing discs and write $\langle D_i\rangle \to \langle E_i\rangle$.

Definition 3.1. Define the cut system complex $\overline{\mathbf{X}}_n$ as follows. The set of all cut systems for a_* forms the vertex set $\overline{\mathbf{X}}_n^0$. Two vertices are connected by a single edge iff they differ by a simple move. Finally, glue faces into every loop of the following form, giving triangular and rectangular faces.



Define the basepoint to be $v_0 = \langle d_1, d_2, \dots, d_n\rangle$ where the d_i are vertical discs below the a_i, see Figure 1. Sometimes it is convenient to think of the a_i and d_i rotated by

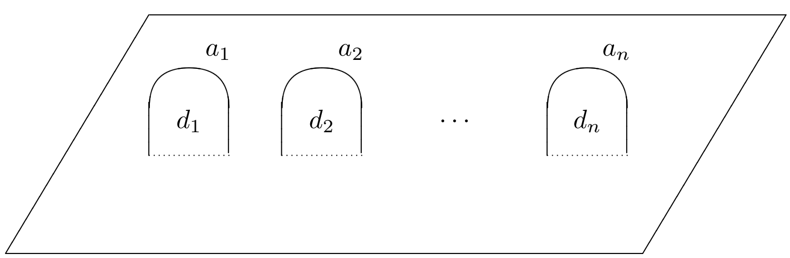


FIGURE 1. The arcs a_i and the discs d_i

a quarter turn.

Before we prove that this complex is simply connected we need the following lemma about substituting one disc for another.

Suppose that $v = \langle D_1, D_2, \dots D_n \rangle$ is a vertex of $\overline{\mathbf{X}}_n$ and that D and D^* are two discs cutting out the arc a_i. We will say that the tuple (v, D, D^*) forms a *valid substitution* if either $D \neq D_i$ for any i, or if there exists some i such that $D = D_i$ and that for all $j \neq i$ we have that $D_j \cap D^* = \emptyset$. In other words if D is in v then (v, D, D^*) forms a valid substitution if there exists an edge $\langle D = D_i \rangle$—$\langle D^* \rangle$. If (v, D, D^*) forms a valid substitution then we can replace D with D^* to get a vertex v^*, ie

$$v^* = \begin{cases} v & \text{if } D_i \neq D, \\ \langle D^* \rangle & \text{if } D_i = D. \end{cases}$$

Similarly, for any edge path P with a choice of discs representing each vertex, we say (P, D, D^*) forms a valid substitution if for each vertex v of P the tuple (v, D, D^*) forms a valid substitution and for each edge (v_i, v_{i+1}) of P there is an edge (v_i^*, v_{i+1}^*). If (P, D, D^*) forms a valid substitution then we can replace each occurrence of D with D^*, ie replace each vertex v with v^*, giving a new path P^*.

Lemma 3.2. *If (P, D, D^*) forms a valid substitution, where $P = (v_1, \dots, v_k)$, then P^* is a path and the loop*

$$\begin{array}{ccc} v_1 & \overset{P}{\text{————}} & v_k \\ | & & | \\ v_1^* & \overset{P^*}{\text{————}} & v_k^* \end{array}$$

is homotopic to a point. Moreover, if P is a loop then so is P^* and they are homotopic as loops.

Proof. Clearly we may assume that D and D^* are not isotopic, otherwise $P = P^*$. Suppose that D and D^* cut out the arc a_i. For each vertex v of P we have that either $v = v^*$ or (v, v^*) is an edge of $\overline{\mathbf{X}}_n$.

For each edge (u, v) in P, where $u = \langle D_j \rangle$ and $v = \langle D'_j \rangle$, we have the following possibilities. If D is not in u nor in v then $(u, v) = (u^*, v^*)$. Otherwise we have two cases depending on whether $i = j$ or not.

If $i = j$ then only one of either u or v contains D. Suppose that $D \in u$, ie $D_j = D$. If $D^* = D'_j$ then $u^* = v^* = v$ and (u, v) is homotopic to (u^*, v^*) in $\overline{\mathbf{X}}_n^1$. Otherwise, if $D^* \neq D_j$, we have the following face of $\overline{\mathbf{X}}_n$.

$$\begin{array}{ccc} u & \overset{P}{\text{————}} & v = v^* \\ | & \diagup_{P^*} & \\ u^* & & \end{array}$$

If $i \neq j$ the we have the following face of $\overline{\mathbf{X}}_n$.

$$\begin{array}{ccc} \langle D, D_j \rangle & \overset{P}{\text{————}} & \langle D, D'_j \rangle \\ | & & | \\ \langle D^*, D_j \rangle & \overset{P^*}{\text{————}} & \langle D^*, D'_j \rangle \end{array}$$

In either case there is a homotopy from (u,v) to (u^*,v^*) that agrees with the homotopies between the vertices of P and P^*. Therefore P is homotopic to P^*. □

Theorem 3.3. *The complex $\overline{\mathbf{X}}_n$ is connected and simply connected.*

Proof. It suffices to show that any loop is homotopic to the constant loop at v_0. Given a loop in $\overline{\mathbf{X}}_n$ it is homotopic to an edge path P. Now choose discs to represent each vertex of P. We shall write $D \in P$ if D is one of the discs chosen as a representative of some vertex of P.

Claim. *The path P is homotopic to a path whose vertices admit representative discs which intersect the discs $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$.*

Assuming that the intersection of the discs $D \in P$ with $d_1 \cup d_2 \cup \ldots \cup d_n$ isn't only $a_1, a_2, \ldots, a_n$ we can carry out the following procedure.

For some i the union of the discs in P intersects d_i in a non-empty collection of arcs. Pick an arc α of this intersection that is lowest in the sense that it doesn't separate the entirety of any other arc from $\partial H^3 \cap d_i$. For example, see Figure 2 where α and γ are lowest but β is not.

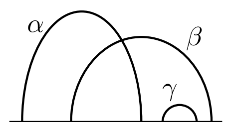


FIGURE 2. Lowest arcs α and γ

The arc α comes from some $D \in P$. Now cut D along α, discard the section not containing a_i and glue in a disc parallel to d_i. This results in a new disc D^* whose intersection with d_i contains at least one less arc.

Any disc $E \in P$ for which $E \cap D = a_j$ or $\emptyset$ also has $E \cap D^* = a_j$ or $\emptyset$ respectively; if not E must intersect D^* in the section parallel to d_i and this contradicts the condition that α is a lowest arc. Therefore the triple (P, D, D^*) form a valid substitution and, by Lemma 3.2, we can replace D with D^* to get a new homotopic loop P^*.

We now have a homotopic loop P^* that has fewer intersections with $d_1 \cup d_2 \cup \ldots \cup d_n$. So by induction on the number of intersections we have proved the claim.

So we may assume that the path P meets $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$. Therefore, for each $D \in P$ cutting out the arc a_i, the triple (P, D, d_i) forms a valid substitution and so by in turn replacing each $D \in P$ with d_i we see that P is homotopic to the constant path v_0. The connectedness of $\overline{\mathbf{X}}_n$ follows by taking P to be a constant loop. □

Up to homotopy the group $\mathbf{H}_{2n}$ acts on (H^3, a_*) by homeomorphisms, therefore it takes cut systems to cut systems. The edges and faces of $\overline{\mathbf{X}}_n$ are determined by the intersections of pairs of discs, hence this action on $\overline{\mathbf{X}}_n^0$ extends to a cellular action on $\overline{\mathbf{X}}_n$.

Theorem 3.4. *The action of $\mathbf{H}_{2n}$ on $\overline{\mathbf{X}}_n^0$ is transitive.*

Proof. Given a vertex $\langle D_1, D_2, \dots, D_n\rangle$ of $\overline{\mathbf{X}}_n$, if we take each i in turn and look at the intersection of D_i with ∂H^3. We see that this defines a path from one end of a_i to the other. If we now move one end around this path until it is close to the other and then move it straight back to its starting point we have an element of $\mathbf{H}_{2n}$ that moves D_i to d_i. Combining all of these we see that $\langle D_1, D_2, \dots, D_n\rangle$ is in the orbit of v_0, ie the action is transitive on $\overline{\mathbf{X}}_n^0$. □

4. The complex $\mathbf{X}_n$

We now construct a subcomplex $\mathbf{X}_n$ of $\overline{\mathbf{X}}_n$ with the same vertex set but with fewer edges and faces.

Given an edge $e = (\langle D\rangle, \langle D'\rangle)$ of $\overline{\mathbf{X}}_n$ define its length, $l(e)$, to be the number of arcs underneath $D \cup D'$. In other words, since $H^3 \setminus D \cup D'$ has two components, one bounded and one unbounded, we can define the length as follows

$$l(e) = \#\{i \mid a_i \text{ is contained in the bounded component of } H^3 \setminus D \cup D'\}.$$

Given two edges e and e' with the same length there exists an element of $\mathbf{H}_{2n}$ taking e to e'.

We will say that a rectangle $(\langle D, E\rangle, \langle D', E\rangle, \langle D', E'\rangle, \langle D, E'\rangle)$ is *nested* if $E \cup E'$ lies in the bounded component of $H^3 \setminus D \cup D'$ or vice versa, ie if one pair of changing discs lies underneath the other.

For $i \le j$ let $\mathcal{T}_{ij}$ denote the subcomplex consisting of all triangular faces of $\overline{\mathbf{X}}_n$ with shortest two edges of length i and j. Note, this implies that the remaining edge has length $i+j$. Given a rectangular face of $\overline{\mathbf{X}}_n$ we have two cases depending on whether it is nested or not. Let $\mathcal{R}_{ij}$ denote the subcomplex consisting of all rectangular nested faces with inner edge of length i and outer edge of length j. For $i \le j$ let $\mathcal{S}_{ij}$ denote the subcomplex consisting of all non-nested rectangular faces with edges of length i and j.

Definition 4.1. Let $\mathbf{X}_n$ be the subcomplex of $\overline{\mathbf{X}}_n$ with the same vertex set, all edges of length 1 and 2 and all faces from $\mathcal{R}_{12}$, $\mathcal{S}_{11}$ and $\mathcal{T}_{11}$, ie $\mathbf{X}_n = \mathcal{R}_{12} \cup \mathcal{S}_{11} \cup \mathcal{T}_{11}$. As the length of an edge is invariant under the action of $\mathbf{H}_{2n}$ on $\overline{\mathbf{X}}_n$ this action preserves each $\mathcal{T}_{ij}$, $\mathcal{R}_{ij}$ and $\mathcal{S}_{ij}$ and so preserves $\mathbf{X}_n$.

A vertex $v = \langle D_1, \dots, D_n\rangle$ is completely determined by the intersection of the discs D_i with ∂H^3. Using this we can define the vertices x_i for $0 \le i \le n-1$, y_{ij} for $0 \le i \le n-2$ and $j = 0$ or $i < j \le n-1$ and z_{ij} for $0 \le i, j, i+j \le n-2$ as in Figure 3. So we have $l(v_0, x_i) = i$, $l(v_0, y_{0j}) = j$, $l(v_0, y_{i0}) = i$, $l(v_0, z_{i0}) = i$ and $l(v_0, z_{0j}) = j$. Note, there is some redundancy in this notation, ie $x_i = y_{0i}$ and $x_0 = y_{00} = z_{00} = v_0$.

We now define the faces $R_{ij} \in \mathcal{R}_{ij}$, $S_{ij} \in \mathcal{S}_{ij}$, $T_{ij} \in \mathcal{T}_{ij}$ of $\overline{\mathbf{X}}_n$ as follows.

$$R_{ij} = \begin{array}{ccc} y_{00} & \text{——} & y_{i0} \\ | & & | \\ y_{0j} & \text{——} & y_{ij} \end{array} \qquad S_{ij} = \begin{array}{ccc} z_{00} & \text{——} & z_{i0} \\ | & & | \\ z_{0j} & \text{——} & z_{ij} \end{array} \qquad T_{ij} = \begin{array}{ccc} x_0 & \text{——} & x_i \\ & \diagdown \;\; \diagup & \\ & x_{i+j} & \end{array}$$

For every face in $\overline{\mathbf{X}}_n$ there is an element of $\mathbf{H}_{2n}$ taking it to one of these representatives.

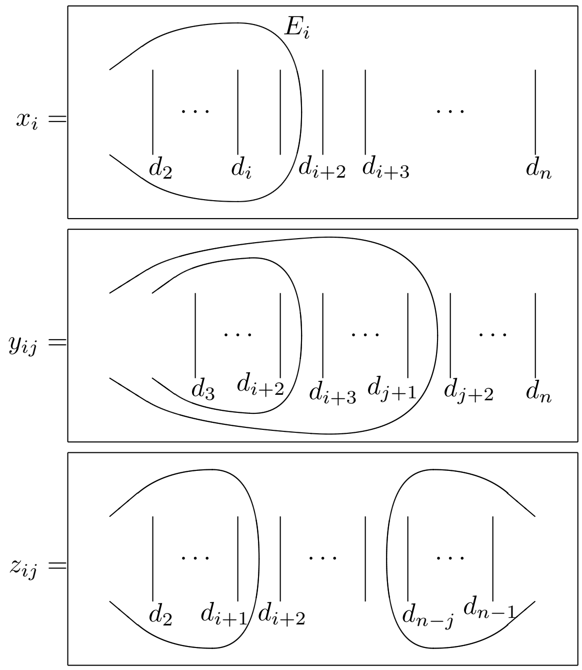


FIGURE 3. The vertices x_i, y_{ij} and z_{ij}

Theorem 4.2. *The complex $\mathbf{X}_n$ is simply connected.*

Proof. Figure 4 shows that the boundary of each of the faces R_{ij} for $1 < i < j$ and S_{ij} for $1 < i, j$ can be expressed as the boundary of a union of faces with shorter edges. The first column shows how to replace faces where the first index is not 1. Then the second column can be used to reduce the second index to either 2 or 1 respectively.

As each of the rectangular faces can be moved to one of R_{ij} or S_{ij} by some element of $\mathbf{H}_{2n}$ it follows that every loop in $\mathbf{X}_n$ is null-homotopic in

$$\mathcal{R}_{12} \cup \mathcal{S}_{11} \cup \bigcup_{1 \le i \le j \le n} \mathcal{T}_{ij}.$$

Let the E_i be the discs as shown in Figure 3, ie $x_i = \langle E_i, d_2, d_3, \ldots, d_n \rangle$. For $j > 2$ let A_j the be full subcomplex of $\overline{\mathbf{X}}_n$ containing all the vertices "between" x_0 and x_j, ie

$$A_j^0 = \{\langle D, d_2, d_3, \ldots, d_n \rangle \in \overline{\mathbf{X}}_n^0 \\ \mid D \neq d_0 \text{ or } E_j, \text{interior of } D \subset \text{bounded component of } H^3 \setminus E_0 \cup E_j\}.$$

Choose x_1 as a base point of A_j.

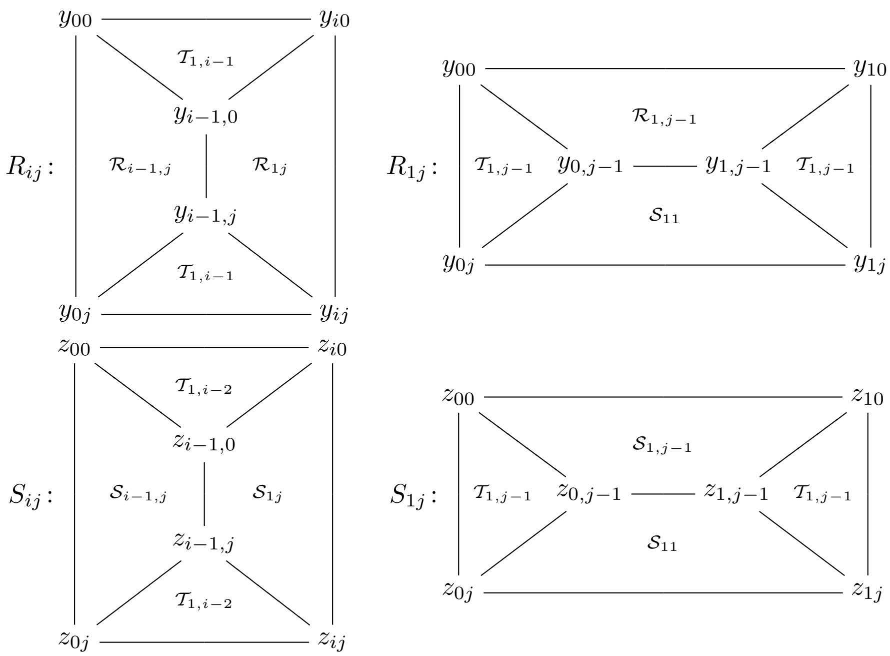
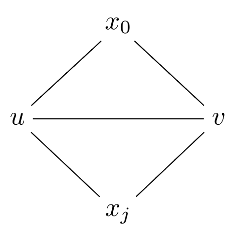


FIGURE 4. Decomposing rectangular faces

For every edge (u, v) of A_j we have the following two triangles in $\overline{\mathbf{X}}_n$. Note that all edges have length less than j.



Lemma 4.3. *The subcomplex A_j is path connected.*

Proof. Given a vertex $v = \langle D, d_2, \dots d_n \rangle \in A_j^0$. First suppose that for some $2 < i \leq j$ there exists a path γ on ∂H^3 from d_i to E_j such that γ does not cross E_1, D or d_l for $l \neq i$. Let D' be a disc parallel to E_j except in a neighbourhood of γ where we glue in the boundary of a neighbourhood of $\gamma \cup d_i$. Then there is a path (v, v', x_1) in A_j where $v' = \langle D' \rangle$. See Figure 5.

Now suppose that no such path exists on ∂H^3. Each vertex $u = \langle D_u \rangle$ of A_j partitions the set $\{d_2, d_3, \dots d_{j+1}\}$ into two non-empty subsets. The first containing those discs that are between d_1 and D_u, the second those between D_u and E_j. (If one of these sets were empty then we would have that either $D_u = d_1$ or $D_u = E_j$.) As $j > 2$ at least one of these sets contains more than one disc. Choose an $i \neq 1$ such that d_i is in this set.

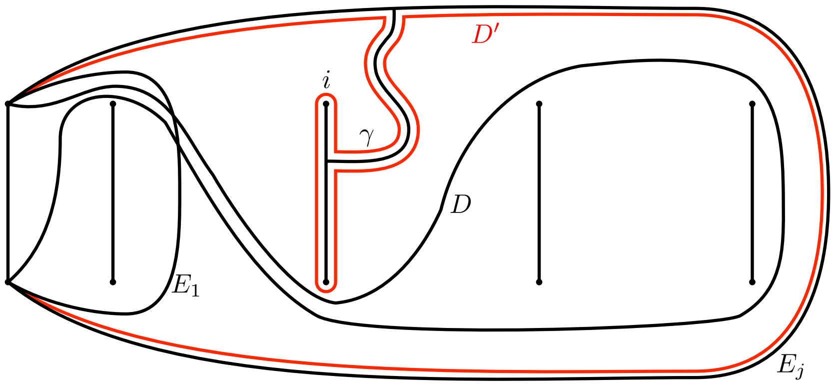
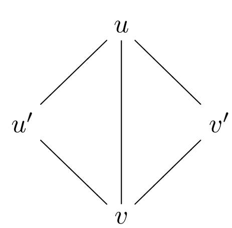


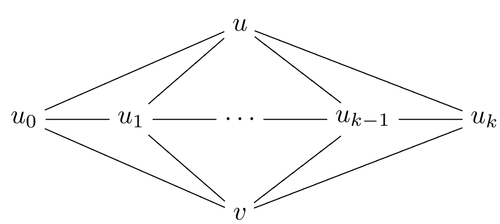
FIGURE 5. Tunnelling along γ

Now draw a path γ on ∂H^3 from d_i to E_j that doesn't intersect d_l for $l = 3, \dots, j$ or E_1 and only intersects D transversely. Starting at d_i move along γ and label the successive points of $\gamma \cap D$ as $p_1, p_2, \dots, p_k$. Now we can construct a sequence of discs $D = D^0, D^1, \dots, D^k$ where each D^{l+1} is parallel to D^l except in a neighbourhood of p_{l+1} where we glue in the boundary of a sufficiently small neighbourhood of the disc d_i and the segment of γ up to p_{l+1}. With each successive D^l the disc d_i moves from one side of the partition to the other. At each step neither side of the partition is empty so $\langle D^l \rangle$ is a vertex of A_j. This gives a path $(v = \langle D^0 \rangle, \langle D^1 \rangle, \dots, \langle D^k \rangle)$ in A_j. Now, $\langle D^k \rangle$ satisfies the hypothesis above, therefore this path can be continued to the base point x_1. □

We can now complete the proof of Theorem 4.2. So far we have shown that any loop in $\mathbf{X}_n$ is the boundary of a union of faces in $\mathcal{R}_{12} \cup \mathcal{S}_{11} \cup \bigcup_{1 \leq i \leq j \leq n} \mathcal{T}_{ij}$. For a given loop take an edge (u, v) of maximal length j in this union. If $j > 2$ then the faces on either side of (u, v) must be triangular with the remaining edges of length less than j. So we have the following situation for some $u', v' \in \overline{\mathbf{X}}_n^0$.



By Lemma 4.3 we can replace these two triangles with the following.



Where $u_0 = u'$ and $u_k = v'$. Here each edge has length less than j. Therefore all edges of length greater that 2 can be replaced and so the loop is null-homotopic in $\mathbf{X}_n$. □

5. Calculating the presentation

By Section 4 we have an $\mathbf{H}_{2n}$–action on a simply connected cellular complex. So we can now follow the method given in Section 2.

Using the fact that $\mathbf{H}_{2n}$ is a subgroup of $\mathbf{B}_{2n}$, we can define the following elements of $\mathbf{H}_{2n}$ in terms of $\sigma_1, \ldots, \sigma_{2n-1}$ the generators of $\mathbf{B}_{2n}$.

$$\begin{aligned}
r_1 &= \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} && \\
r_2 &= \sigma_4\sigma_3\sigma_2\sigma_1\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1} && \\
s_i &= \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i} && \text{for } i \in \{1, \ldots, n-1\} \\
t_i &= \sigma_{2i-1} && \text{for } i \in \{1, \ldots, n\}
\end{aligned}$$

So r_1 is the first arc passing through the second, r_2 is the first two arcs passing through the third, s_i is the ith and $i+1$st arcs crossing and t_i is the ith arc performing a half twist. Subsequently we will prove that these generate $\mathbf{H}_{2n}$.

Proposition 5.1. *The stabiliser of the vertex v_0 is isomorphic to the framed braid group and hence has a presentation $\langle S_0 \mid R_0 \rangle$ where*

$$\begin{aligned}
S_0 &= \{s_1, s_2, \ldots, s_{n-1},\ t_1, t_2, \ldots, t_n\} \\
R_0 &= \{ \quad s_is_j = s_js_i \quad \textit{for } |i-j| > 1, \\
&\qquad s_is_js_i = s_js_is_j \quad \textit{for } |i-j| = 1, \\
&\qquad t_it_j = t_jt_i \quad \textit{for all } i, j, \\
&\qquad s_it_j = t_js_i \quad \textit{if } j \notin \{i, i+1\}, \\
&\qquad s_it_j = t_ks_i \quad \textit{if } \{i, i+1\} = \{j, k\} \quad \}
\end{aligned}$$

Proof. If we restrict to ∂H^3, elements of $\mathbf{H}_{2n}$ can be thought of as motions of the end points of the a_i. For elements of the vertex stabiliser this motion moves the $d_i \cap \partial H^3$ among themselves, ie this is the fundamental group of configurations of n line segments in the plain, the framed braid group. □

We have two edge orbits, one consisting of edges of length 1 and the other consisting of edges of length 2. Note that our choice of r_1 and r_2 mean that

$$(v_0, v_0 \cdot r_1) \in l^{-1}(1)$$
$$(v_0, v_0 \cdot r_2) \in l^{-1}(2).$$

For $i = 1,\ 2$, let I_i denote the stabiliser of the edge $(v_0, v_0 \cdot r_i)$, ie the subgroup of all elements that fix both v_0 and $v_0 \cdot r_i$.

Proposition 5.2. *The subgroups I_1 and I_2 are generated as follows.*

$$I_1 = \langle t_2, t_3, \ldots, t_n,\ s_3, s_4, \ldots, s_{n-1},\ s_1s_1t_1t_1,\ s_2s_1s_1s_2 \rangle$$
$$I_2 = \langle t_2, t_3, \ldots, t_n,\ s_2,\ s_4, s_5, \ldots, s_{n-1},\ s_1s_2s_2s_1t_1t_1,\ s_3s_2s_1s_1s_2s_3 \rangle$$

Proof. For I_1 $[I_2]$ the motion of the d_i outside of $d_1 \cup E_2$ $[d_1 \cup E_3]$ is generated by $t_3, t_4, \dots, t_n, s_3, s_4, \dots, s_{n-1}$ and $s_2s_1s_1s_2$ $[t_4, t_5, \dots, t_n,\ s_4, s_5, \dots, s_{n-1}$ and $s_3s_2s_1s_1s_2s_3]$, the motion of the d_i inside $d_1 \cup E_2$ $[d_1 \cup E_3]$ is generated by t_2 $[t_2, t_3, s_2]$ and the motion of $d_1 \cup E_2$ $[d_1 \cup E_3]$ is generated by $s_1s_1t_1t_1$ $[s_1s_2s_2s_1t_1t_1]$. □

We are now ready to calculate relations for R_1, R_2 and R_3. The following relations are easily verifiable, in fact most of them take place in $\mathbf{B}_8$.

The R_1 relations. To calculate the R_1 relations we have to find, for each edge orbit representative $(v_0, v_0 \cdot r_i)$ and each generator t of I_i, a word h in S_0 such that $r_itr_i^{-1} = h$. One possibility is the following.

(R_11) $$r_1t_2r_1^{-1} = t_1$$
(R_12) $$r_1t_kr_1^{-1} = t_k \qquad \text{for } k > 2$$
(R_13) $$r_1s_kr_1^{-1} = s_k \qquad \text{for } k > 2$$
(R_14) $$r_1s_1s_1t_1t_1r_1^{-1} = s_1s_1t_2t_2$$
(R_15) $$r_1s_2s_1s_1s_2r_1^{-1} = s_2s_1s_1s_2$$

(R_16) $$r_2t_2r_2^{-1} = t_1$$
(R_17) $$r_2t_3r_2^{-1} = t_2$$
(R_18) $$r_2t_kr_2^{-1} = t_k \qquad \text{for } k > 3$$
(R_19) $$r_2s_2r_2^{-1} = s_1$$
(R_110) $$r_2s_kr_2^{-1} = s_k \qquad \text{for } k > 3$$
(R_111) $$r_2s_1s_2s_2s_1t_1t_1r_2^{-1} = s_2s_1s_1s_2t_3t_3$$
(R_112) $$r_2s_3s_2s_1s_1s_2s_3r_2^{-1} = s_3s_2s_1s_1s_2s_3$$

The R_2 relations. To calculate the R_2 relations we need to find, for each edge orbit representative $(v_0, v_0 \cdot r_i)$, an h-product $r_ih\ r_i$ for the path $(v_0, v_0 \cdot r_i, v_0)$ and a word h' in S_0 such that $r_ih\ r_i = h'$.

(R_21) $$r_1t_1s_1\ r_1 = s_1t_1$$
(R_22) $$r_2s_1t_2s_2\ r_2 = s_2s_1t_1$$

The R_3 relations. To calculate the R_3 relations we need to find, for each edge orbit, an h-product representing the boundary of a face in the orbit and an equivalent word in S_0. The following are such relations for the $\mathcal{S}_{11}$, $\mathcal{R}_{12}$ and $\mathcal{T}_{11}$ orbits respectively.

(R_31) $$\begin{aligned} &r_1s_1s_2s_3s_1s_2\ r_1s_1s_2s_3s_1s_2t_2t_4\ r_1s_2s_3s_1s_2\ r_1 \\ &\quad = s_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_2s_3s_1s_2t_1t_3 \end{aligned}$$
(R_32) $$r_1\ r_2s_1s_2s_1t_2t_3\ r_1\ r_2 = s_2s_1s_2t_1t_2$$
(R_33) $$r_2s_1t_2\ r_1s_2s_1\ r_1 = s_1s_2s_1t_1$$

If we use a different set of generators, similar to those found by Hilden, then we can get a more braid like presentation. Let $p_i = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}$ for $1 \le i < n$. So p_i is the ith arc passing under the $i+1$st arc, see Figure 7.

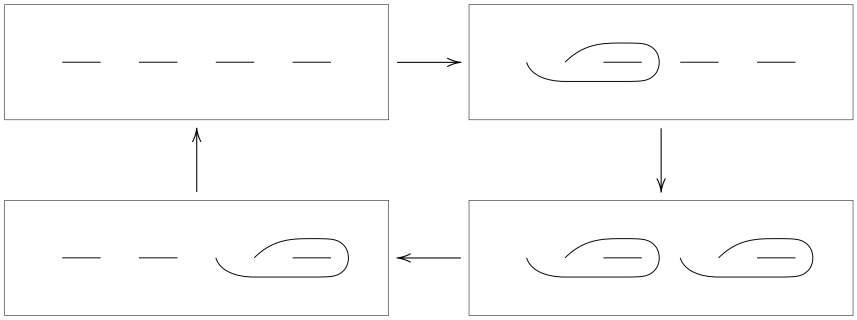


FIGURE 6. The path given by the h-product on the LHS of $(R_3 1)$

Theorem 5.3. *The group* $\mathbf{H}_{2n}$ *has a presentation with generators* p_i*,* s_j *and* t_k *for* $1 \leq i, j < n$ *and* $1 \leq k \leq n$ *and the following relations.*

(P1)	$p_i p_j = p_j p_i$	*for* $\lvert i-j \rvert > 1$
(P2)	$p_i p_j p_i = p_j p_i p_j$	*for* $\lvert i-j \rvert = 1$
(P3)	$s_i s_j = s_j s_i$	*for* $\lvert i-j \rvert > 1$
(P4)	$s_i s_j s_i = s_j s_i s_j$	*for* $\lvert i-j \rvert = 1$
(P5)	$p_i s_j = s_j p_i$	*for* $\lvert i-j \rvert > 1$
(P6)	$p_i s_{i+1} s_i = s_{i+1} s_i p_{i+1}$	
(P7)	$p_{i+1} p_i s_{i+1} = s_i p_{i+1} p_i$	
(P8)	$p_{i+1} s_i s_{i+1} = s_i s_{i+1} p_i$	
(P9)	$p_i t_i s_i p_i = s_i t_i$	
(P10)	$p_i t_j = t_j p_i$	*for* $j \neq i$*, or* $i+1$
(P11)	$p_i t_{i+1} = t_i p_i$	
(P12)	$s_i t_j = t_j s_i$	*if* $j \neq i$ *or* $i+1$
(P13)	$s_i t_j = t_k s_i$	*if* $\{i, i+1\} = \{j, k\}$
(P14)	$t_i t_j = t_j t_i$	*for* $1 \leq i, j \leq n$

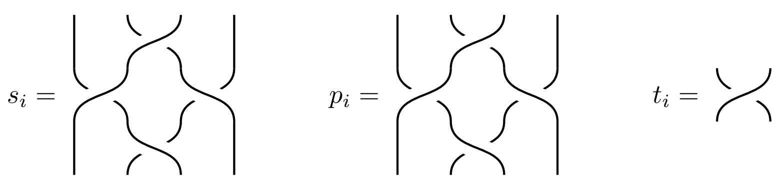


FIGURE 7. Generators of $\mathbf{H}_{2n}$

These generators and relations can be represented pictorially as in Figure 8 and Figure 9.

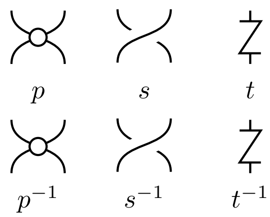


FIGURE 8. Pictorial representation of the p, s, t and their inverses

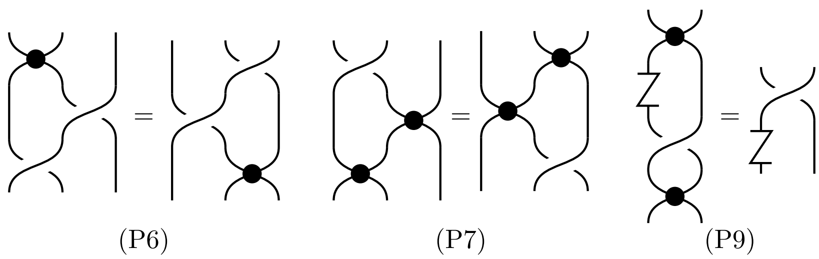


FIGURE 9. Pictorial representation of (P6), (P7) and (P9)

Proof. Since $\mathbf{H}_{2n}$ is a subgroup of the braid group it is easy to check that these relations all hold. So it remains to prove that each of the relations in R_0, R_1, R_2 and R_3 can be deduced from (P1)–(P14) using the fact that $r_1 = p_1$ and $r_2 = p_2 p_1$. First note that R_0 is a subset of these relations. The relations ($R_1$1), ($R_1$2), ($R_1$3) and ($R_2$1) follow directly from (P11), (P10), (P5) and (P9) respectively. The remaining relations can be deduced as follows. Some of these relations are quite long and are perhaps better understood using pictorial representations. For the longest, ($R_3$1), see Figure 10 for a pictorial version.

($R_1$4):
$$\begin{aligned} r_1 s_1 s_1 t_1 t_1 r_1^{-1} &= p_1 \underline{s_1 s_1 t_1} t_1 p_1^{-1} && \text{(P13)}^2 \\ &= \underline{p_1 t_1 s_1} s_1 t_1 p_1^{-1} && \text{(P9)} \\ &= s_1 t_1 \underline{p_1^{-1} s_1 t_1 p_1^{-1}} && \text{(P9)} \\ &= s_1 \underline{t_1 t_1 s_1} && \text{(P13)}^2 \\ &= s_1 s_1 t_2 t_2 \end{aligned}$$

($R_1$5):
$$\begin{aligned} r_1 s_2 s_1 s_1 s_2 r_1^{-1} &= \underline{p_1 s_2 s_1} s_1 s_2 p_1^{-1} && \text{(P6)} \\ &= s_2 s_1 \underline{p_2 s_1 s_2} p_1^{-1} && \text{(P8)} \\ &= s_2 s_1 s_1 s_2 \end{aligned}$$

($R_1$6):
$$\begin{aligned} r_2 t_2 r_2^{-1} &= p_2 \underline{p_1 t_2 p_1^{-1}} p_2^{-1} && \text{(P11)} \\ &= \underline{p_2 t_1 p_2^{-1}} && \text{(P10)} \\ &= t_1 \end{aligned}$$

($R_1$7):
$$\begin{aligned} r_2 t_3 r_2^{-1} &= p_2 \underline{p_1 t_3 p_1^{-1}} p_2^{-1} && \text{(P10)} \\ &= \underline{p_2 t_3 p_2^{-1}} && \text{(P11)} \\ &= t_2 \end{aligned}$$

($R_1$8):
$$\begin{aligned} r_2 t_k r_2^{-1} &= p_2 \underline{p_1 t_k p_1^{-1}} p_2^{-1} && \text{(P10)} \end{aligned}$$

$$\begin{aligned} &= \underline{p_2 t_k} p_2^{-1} && \text{(P10)} \\ &= t_k \end{aligned}$$

$$(R_1 9): \quad \begin{aligned} r_2 s_2 r_2^{-1} &= \underline{p_2 p_1 s_2} p_1^{-1} p_2^{-1} && \text{(P7)} \\ &= s_1 \end{aligned}$$

$$(R_1 10): \quad \begin{aligned} r_2 s_k r_2^{-1} &= \underline{p_2 p_1 s_k} p_1^{-1} p_2^{-1} && \text{(P5)}^2 \\ &= s_k \end{aligned}$$

To deduce $(R_1 11)$ we make use of the following deduction.

$$(\star) \quad \begin{aligned} & \underline{p_2 p_1} s_1 s_2 t_3 p_2 p_1 && \text{(P7)} \\ &= s_1^{-1} p_2 p_1 \underline{s_2 s_1 s_2} t_3 p_2 p_1 && \text{(P4)} \\ &= s_1^{-1} p_2 p_1 \underline{s_1 s_2 s_1 t_3} p_2 p_1 && \text{(P12)(P13)}^2 \\ &= s_1^{-1} p_2 p_1 t_1 s_1 \underline{s_2 s_1 p_2} p_1 && \text{(P6)} \\ &= s_1^{-1} p_2 \underline{p_1 t_1 s_1 p_1} s_2 s_1 p_1 && \text{(P9)} \\ &= \underline{s_1^{-1} p_2 s_1} t_1 s_2 s_1 p_1 && \text{(P8)} \\ &= s_2 p_1 s_2^{-1} \underline{t_1 s_2} s_1 p_1 && \text{(P12)} \\ &= s_2 \underline{p_1 t_1 s_1 p_1} && \text{(P9)} \\ &= s_2 s_1 t_1 \end{aligned}$$

$$(R_1 11): \quad \begin{aligned} r_2 s_1 s_2 s_2 s_1 t_1 t_1 r_2^{-1} &= p_2 p_1 s_1 s_2 \underline{s_2 s_1 t_1} t_1 p_1^{-1} p_2^{-1} && \text{(P13)}^2 \\ &= p_2 p_1 s_1 s_2 t_3 \underline{s_2 s_1 t_1 p_1^{-1} p_2^{-1}} && (\star) \\ &= \underline{p_2 p_1 s_1 s_2 t_3 p_2 p_1} s_1 s_2 t_3 && (\star) \\ &= s_2 s_1 \underline{t_1 s_1 s_2} t_3 && \text{(P13)}^2 \\ &= s_2 s_1 s_1 s_2 t_3 t_3 \end{aligned}$$

$$(R_1 12): \quad \begin{aligned} r_2 s_3 s_2 s_1 s_1 s_2 s_3 r_2^{-1} &= p_2 \underline{p_1 s_3} s_2 s_1 s_1 s_2 s_3 p_1^{-1} p_2^{-1} && \text{(P5)} \\ &= p_2 s_3 \underline{p_1 s_2 s_1} s_1 s_2 s_3 p_1^{-1} p_2^{-1} && \text{(P6)} \\ &= p_2 s_3 s_2 s_1 \underline{p_2 s_1 s_2} s_3 p_1^{-1} p_2^{-1} && \text{(P8)} \\ &= p_2 s_3 s_2 s_1 s_1 s_2 \underline{p_1 s_3} p_1^{-1} p_2^{-1} && \text{(P5)} \\ &= \underline{p_2 s_3 s_2} s_1 s_1 s_2 s_3 p_2^{-1} && \text{(P6)} \\ &= s_3 s_2 \underline{p_3 s_1 s_1} s_2 s_3 p_2^{-1} && \text{(P5)}^2 \\ &= s_3 s_2 s_1 s_1 \underline{p_3 s_2 s_3} p_2^{-1} && \text{(P8)} \\ &= s_3 s_2 s_1 s_1 s_2 s_3 \end{aligned}$$

$$(R_2 2): \quad \begin{aligned} r_2 s_1 t_2 s_2 r_2 &= p_2 p_1 s_1 \underline{t_2 s_2 p_2} p_1 && \text{(P9)} \\ &= p_2 p_1 s_1 p_2^{-1} s_2 t_2 \underline{p_1} && \text{(P9)} \\ &= p_2 p_1 s_1 p_2^{-1} s_2 \underline{t_2 s_1^{-1}} t_1^{-1} p_1^{-1} s_1 t_1 && \text{(P13)} \\ &= p_2 p_1 \underline{s_1 p_2^{-1}} s_2 s_1^{-1} p_1^{-1} s_1 t_1 && \text{(P6)} \\ &= p_2 p_1 s_2^{-1} p_1^{-1} \underline{s_2 s_1 s_2} s_1^{-1} p_1^{-1} s_1 t_1 && \text{(P4)} \\ &= p_2 p_1 s_2^{-1} p_1^{-1} \underline{s_1 s_2 p_1^{-1}} s_1 t_1 && \text{(P8)} \\ &= p_2 p_1 \underline{s_2^{-1} p_1^{-1} p_2^{-1}} s_1 s_2 s_1 t_1 && \text{(P7)} \\ &= s_2 s_1 t_1 \end{aligned}$$

$$(R_3 1): \quad \begin{aligned} & r_1 s_1 s_2 s_3 s_1 s_2 r_1 s_1 s_2 s_3 s_1 s_2 t_2 t_4 r_1 s_2 s_3 s_1 s_2 r_1 \\ &= p_1 s_1 s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 \underline{s_1 s_2 t_2} t_4 p_1 s_2 s_3 s_1 s_2 p_1 && \text{(P13)(P12)} \\ &= p_1 s_1 s_2 s_3 s_1 s_2 p_1 \underline{s_1 s_2 s_3 t_3} s_1 s_2 t_4 p_1 s_2 s_3 s_1 s_2 p_1 && \text{(P13)(P12)}^2 \end{aligned}$$

$$
\begin{aligned}
&= p_1 s_1 s_2 s_3 \underline{s_1 s_2 p_1 t_4} s_1 s_2 s_3 s_1 s_2 t_4 p_1 s_2 s_3 s_1 s_2 p_1 && \text{(P10)(P12)}^2 \\
&= p_1 \underline{s_1 s_2 s_3 t_4} s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 p_1 s_2 s_3 s_1 s_2 p_1 && \text{(P13)}^3 \\
&= p_1 t_1 s_1 s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 \underline{t_4 p_1} s_2 s_3 s_1 s_2 p_1 && \text{(P10)} \\
&= p_1 t_1 s_1 s_2 s_3 s_1 s_2 p_1 s_1 \underline{s_2 s_3 s_1 s_2 p_1} t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P8)}^2 \\
&= p_1 t_1 s_1 s_2 s_3 s_1 s_2 \underline{p_1 s_1 p_3} s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P5)(P1)} \\
&= p_1 t_1 s_1 s_2 \underline{s_3 s_1} s_2 p_3 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P3)} \\
&= p_1 t_1 s_1 s_2 s_1 \underline{s_3 s_2 p_3} p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P6)} \\
&= p_1 t_1 s_1 \underline{s_2 s_1 p_2} s_3 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P6)} \\
&= p_1 t_1 s_1 p_1 s_2 \underline{s_1 s_3} s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P3)} \\
&= \underline{p_1 t_1 s_1 p_1} s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 && \text{(P9)} \\
&= s_1 t_1 s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 \underline{s_2 s_3 s_1 s_2 p_1} && \text{(P8)}^2 \\
&= s_1 t_1 s_2 s_3 s_1 s_2 p_1 \underline{s_1 s_2 s_3 s_1 s_2 t_4} p_3 s_2 s_3 s_1 s_2 && \text{(P12)}^2\text{(P13)}^3 \\
&= s_1 t_1 s_2 s_3 s_1 s_2 p_1 t_1 s_1 s_2 \underline{s_3 s_1} s_2 p_3 s_2 s_3 s_1 s_2 && \text{(P3)} \\
&= s_1 t_1 s_2 s_3 s_1 s_2 p_1 t_1 s_1 \underline{s_2 s_1 s_3 s_2 p_3} s_2 s_3 s_1 s_2 && \text{(P6)}^2 \\
&= s_1 t_1 s_2 s_3 s_1 s_2 \underline{p_1 t_1 s_1 p_1} s_2 s_1 s_3 s_2 s_2 s_3 s_1 s_2 && \text{(P9)} \\
&= s_1 t_1 s_2 s_3 s_1 s_2 s_1 \underline{t_1 s_2 s_1} s_3 s_2 s_2 s_3 s_1 s_2 && \text{(P12)(P13)} \\
&= s_1 t_1 s_2 s_3 s_1 s_2 s_1 s_2 s_1 \underline{t_2 s_3 s_2 s_2} s_3 s_1 s_2 && \text{(P12)(P13)}^2 \\
&= s_1 t_1 s_2 s_3 s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_2 \underline{t_2 s_3 s_1} s_2 && \text{(P12)(P13)} \\
&= s_1 t_1 s_2 s_3 s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_2 s_3 s_1 \underline{t_1 s_2} && \text{(P12)} \\
&= s_1 \underline{t_1 s_2 s_3 s_1 s_2} s_1 s_2 s_1 s_3 s_2 s_2 s_3 s_1 s_2 t_1 && \text{(P12)}^2\text{(P13)}^2 \\
&= s_1 s_2 s_3 s_1 s_2 \underline{t_3 s_1 s_2 s_1} s_3 s_2 s_2 s_3 s_1 s_2 t_1 && \text{(P12)(P13)}^2 \\
&= s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_1 \underline{t_1 s_3 s_2 s_2 s_3 s_1 s_2} t_1 && \text{(P12)}^4\text{(P13)}^2 \\
&= s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_2 s_3 s_1 s_2 \underline{t_3 t_1} && \text{(P14)} \\
&= s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_2 s_3 s_1 s_2 t_1 t_3
\end{aligned}
$$

$$
\begin{aligned}
(R_3 2): \quad r_1 r_2 s_1 s_2 s_1 t_3 t_2 r_1 r_2 &= p_1 p_2 p_1 \underline{s_1 s_2 s_1 t_3} t_2 p_1 p_2 p_1 && \text{(P12)(P13)}^2 \\
&= p_1 p_2 \underline{p_1 t_1 s_1} s_2 s_1 t_2 p_1 p_2 p_1 && \text{(P9)} \\
&= p_1 p_2 s_1 t_1 \underline{p_1^{-1} s_2 s_1} t_2 p_1 p_2 p_1 && \text{(P6)} \\
&= p_1 p_2 s_1 t_1 s_2 s_1 p_2^{-1} t_2 \underline{p_1 p_2 p_1} && \text{(P7)} \\
&= p_1 p_2 s_1 t_1 s_2 s_1 p_2^{-1} \underline{t_2 p_2} p_1 p_2 && \text{(P11)} \\
&= p_1 p_2 s_1 \underline{t_1 s_2} s_1 \underline{t_3 p_1} p_2 && \text{(P12)(P10)} \\
&= p_1 p_2 s_1 s_2 \underline{t_1 s_1 p_1} t_3 p_2 && \text{(P9)} \\
&= p_1 p_2 \underline{s_1 s_2 p_1^{-1}} s_1 t_1 t_3 p_2 && \text{(P8)} \\
&= p_1 \underline{s_1 s_2 s_1} t_1 t_3 p_2 && \text{(P4)} \\
&= \underline{p_1 s_2 s_1} s_2 t_1 t_3 p_2 && \text{(P6)} \\
&= s_2 s_1 p_2 s_2 \underline{t_1 t_3 p_2} && \text{(P14)(P10)} \\
&= s_2 s_1 \underline{p_2 s_2 t_3 p_2} t_1 && \text{(P9)} \\
&= s_2 s_1 s_2 t_2 t_1
\end{aligned}
$$

$$
\begin{aligned}
(R_3 3): \quad r_2 s_1 t_2 r_1 s_2 s_1 r_1 &= p_2 \underline{p_1 s_1 t_2 p_1} s_2 s_1 p_1 && \text{(P9)} \\
&= p_2 s_1 \underline{t_1 s_2} s_1 p_1 && \text{(P10)} \\
&= \underline{p_2 s_1 s_2} t_1 s_1 p_1 && \text{(P8)} \\
&= s_1 s_2 \underline{p_1 t_1 s_1 p_1} && \text{(P9)} \\
&= s_1 s_2 s_1 t_1
\end{aligned}
$$

□

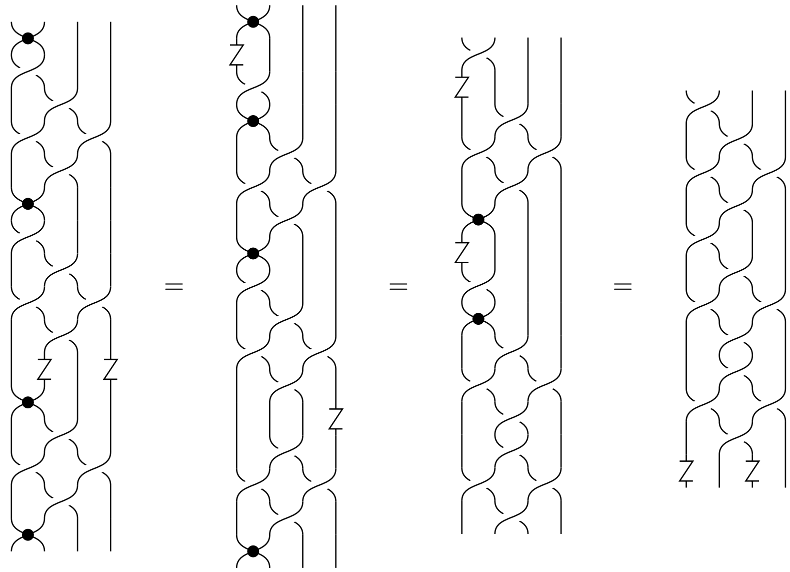


FIGURE 10. Deducing the $(R_3 1)$ relation

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