Normal forms of random braids

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Background

- The positive braid monoid is the monoid given by the following presentation:

$$B_n^+ = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i & (1 \leqslant i < j < n) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i < n-1) \end{array} \right\rangle^+$$

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- $\mathcal{A} := \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ is the set of *atoms*.

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- $k + l$ is the *supremum*

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$$\lambda_i(x) = \begin{cases} x_i & \text{for } i = 1, \ldots, l \\ \mathbf{1} & \text{otherwise} \end{cases} \quad \text{and} \quad \rho_i(x) = \begin{cases} x_{l+1-i} & \text{for } i = 1, \ldots, l \\ \mathbf{1} & \text{otherwise} \end{cases}$$

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where $NF(x) = \Delta^k x_1 x_2 \cdots x_l$. This gives sequences of induced probability measures

$$\lambda_{i*}(\mathsf{Word}_k), \qquad \lambda_{i*}(\mathsf{URB}_k), \qquad \rho_{i*}(\mathsf{Word}_k), \qquad \rho_{i*}(\mathsf{URB}_k)$$

on the symmetric group.

Invariants

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- Finishing set:

$$F(x) := \{\sigma_i \in \mathcal{A} : x \succcurlyeq \sigma_i\}$$

Experiments

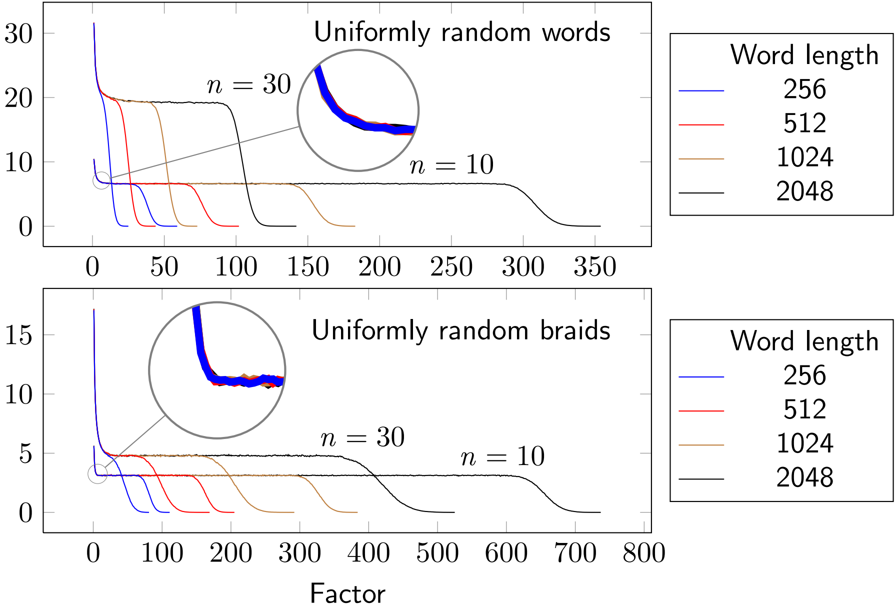
- We constructed and analysed samples of 9999 elements of B_n^+ for each combination of

$$n \in \{5, 10, 15, 20, 25, 30\}$$
$$k \in \{4, 8, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, 512, 1024, 2048\}$$

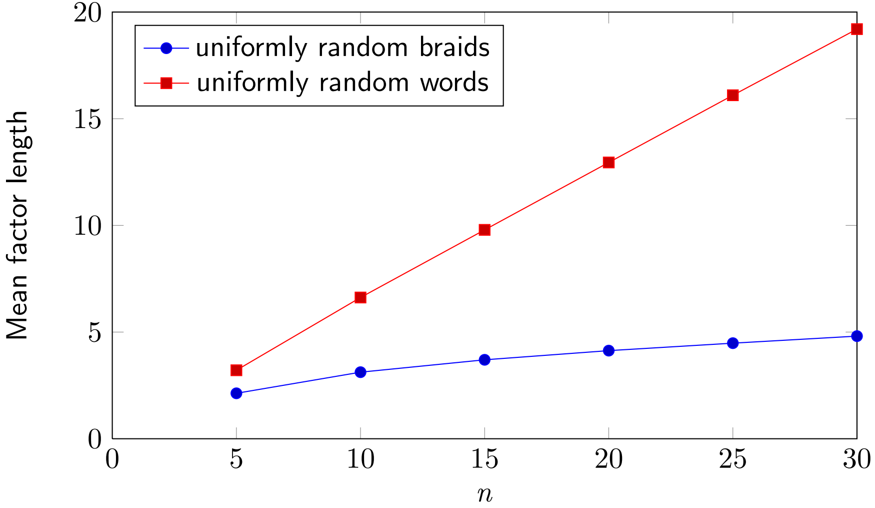
  for both the Word_k and URB_k distributions.

- For Word_k we also analysed samples with a word length of 4096.

Mean factor length

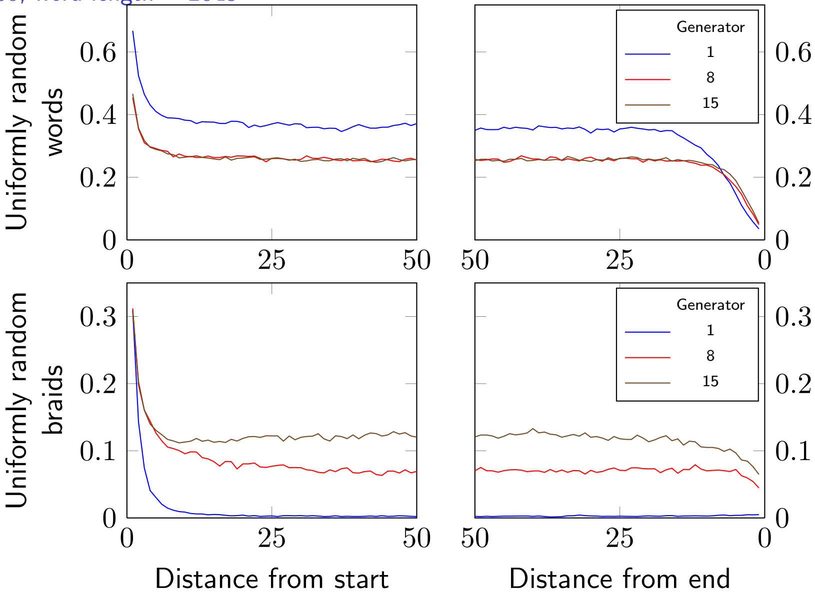


Mean factor length inside stable region.



Relative frequency of a generator being in the starting set

$n = 30$, word length $= 2048$



Conjecture (Stable region)

Consider the braid monoid B_n^+ for any fixed $n \in \mathbb{N}$. For $\mu_k = \mathrm{Word}_k$, respectively $\mu_k = \mathrm{URB}_k$, and for each i, the sequences of probability measures $\lambda_{i*}(\mu_k)$ and $\rho_{i*}(\mu_k)$ on the set of simple elements converge as $k \to \infty$. Moreover, there exists a probability measure Σ on the set of simple elements and constants C and D such that one has

$$\forall i > C \quad \lambda_{i*}(\mu_k) \to \Sigma \text{ as } k \to \infty$$

and

$$\forall i > D \quad \rho_{i*}(\mu_k) \to \Sigma \text{ as } k \to \infty \, .$$

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- Write $\mathcal{L}^{(k)}$ for the subset of words of length k.

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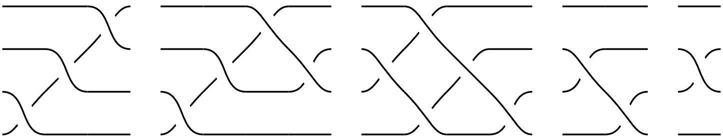

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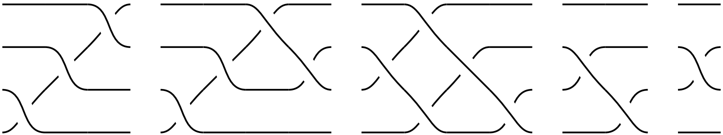

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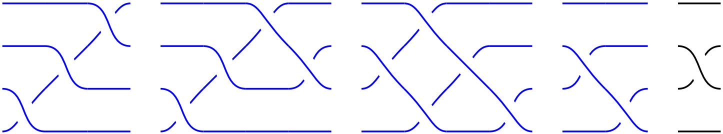

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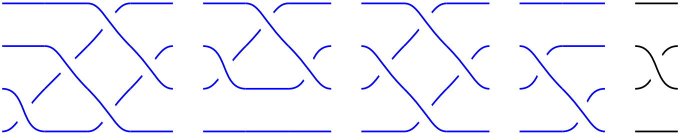

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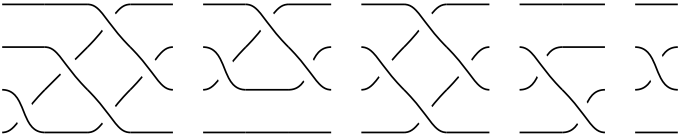

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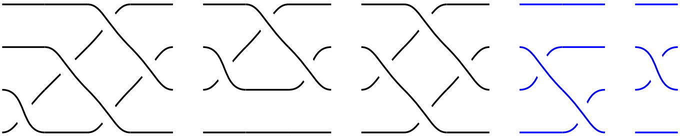

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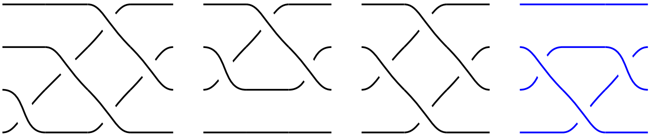

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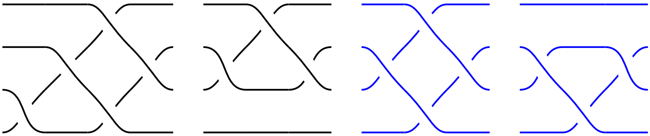

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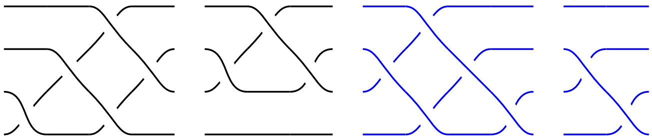

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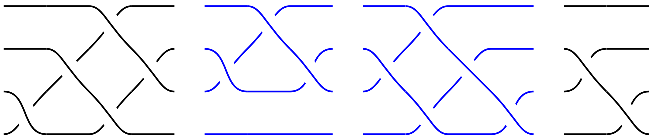

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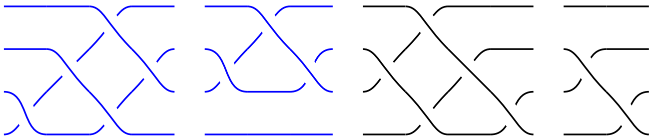

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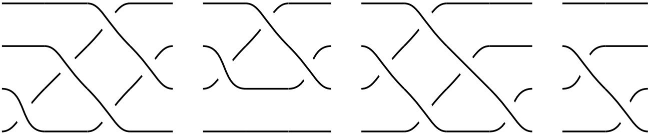

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Penetration distance

The normal form of a random word can be though of as a random process.

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$$\mathrm{pd}(x, y) = \mathrm{cl}(x) - \max\Big\{ i \in \{0, \dots, \mathrm{cl}(x)\} : \\ x\Delta^{-\inf(x)} \wedge \Delta^{i} = xy\Delta^{-\inf(xy)} \wedge \Delta^{i} \Big\}$$

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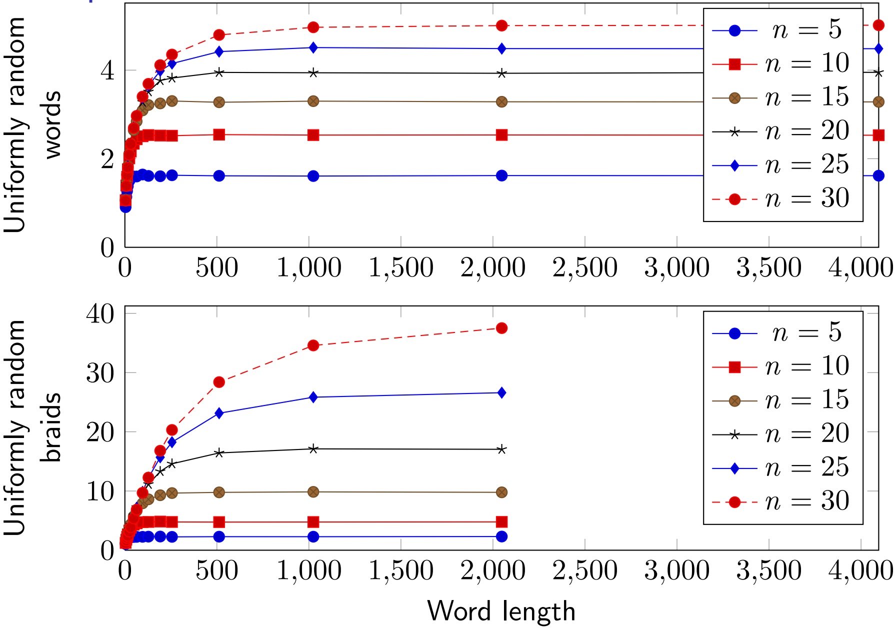
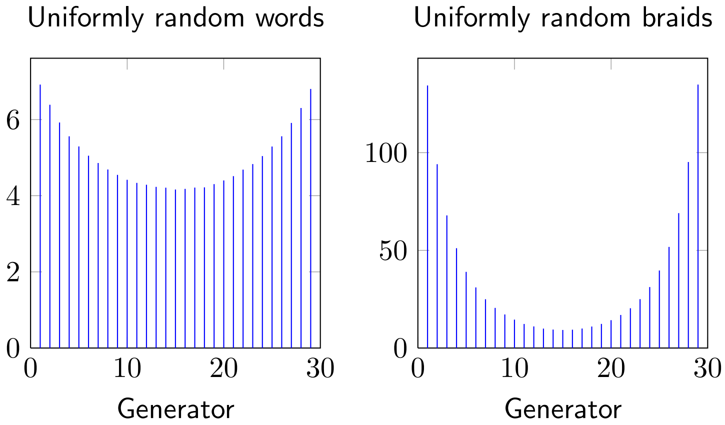
The stable region conjecture suggest that the expected value of pd is bounded.

Mean penetration distance



Mean penetration distance for each generator

$n = 30$, word length $= 2048$



Bounded penetration distance conjecture

Conjecture (Uniformly bounded expected penetration distance)

Consider the braid monoid B_n^+ *for fixed* $n \in \mathbb{N}$*, let* $\mu_{\mathcal{A}}$ *be the uniform probability measure on the set of atoms and, for* $k \in \mathbb{N}$*, let* $\mu_k \in \{\mathsf{Word}_k, \mathsf{URB}_k\}$*. Then there exists* C *such that for all* $k \in \mathbb{N}$*, we have*

$$\mathbf{E}_{\mu_k \times \mu_{\mathcal{A}}}[\mathrm{pd}] < C \ .$$

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Corollary. There's a linear expected time algorithm to compute the normal form of a random word

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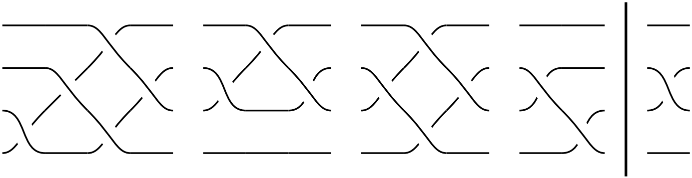

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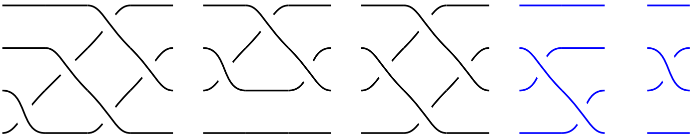

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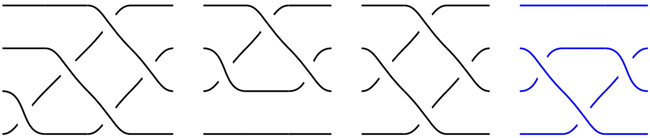

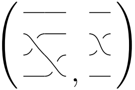

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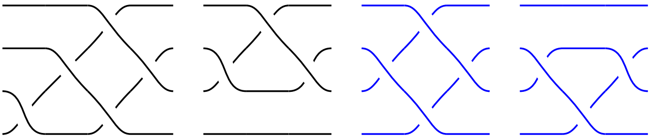

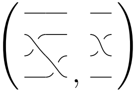

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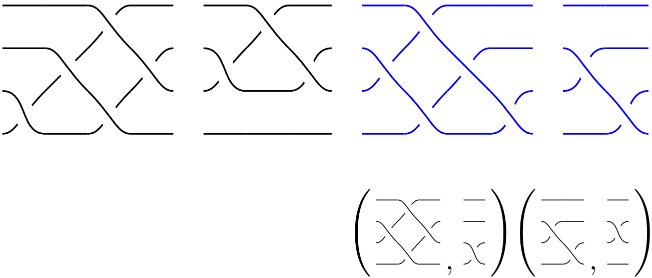

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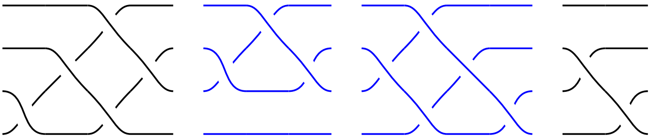

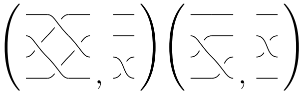

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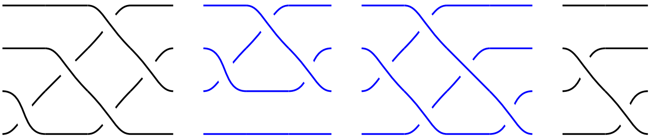

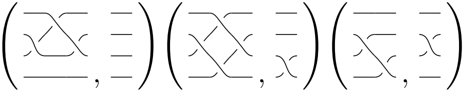

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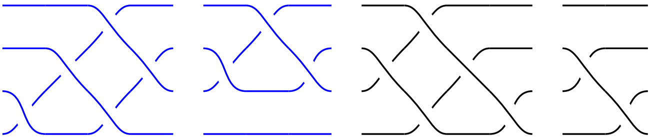

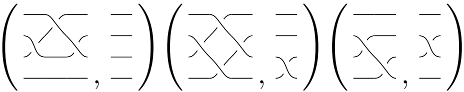

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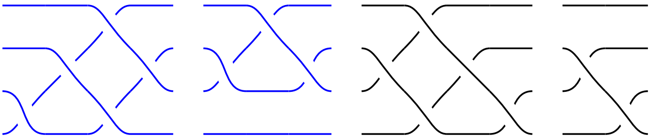

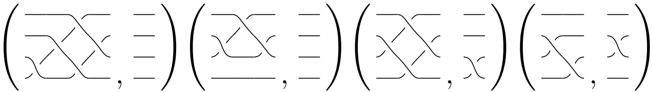

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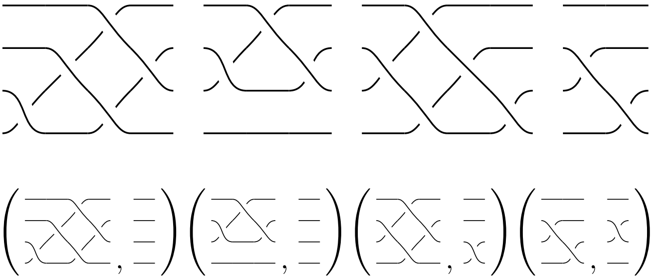

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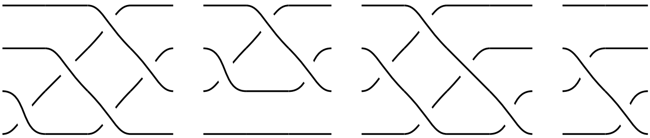

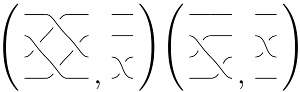

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Proposition

There exist constants α, β, p, $q \geqslant 0$ such that

$$|\mathsf{PSeq}_k| \in \Theta(k^p \alpha^k) \qquad \textit{and} \qquad |\mathcal{L}^{(k)}| \in \Theta(k^q \beta^k).$$

α and β are the exponential growth rates of $|\mathsf{PSeq}_k|$ and $|\mathcal{L}^{(k)}|$.

Criterion for bounded expected pd

Theorem

Let ν_k be the uniform probability measure on $\mathcal{L}^{(k)}$. If $\alpha < \beta$ then the expected value $\mathbf{E}_{\nu_k \times \mu_{\mathcal{A}}}[\mathrm{pd}]$ of the penetration distance with respect to $\nu_k \times \mu_{\mathcal{A}}$ is uniformly bounded (that is, the bound does not depend on k).

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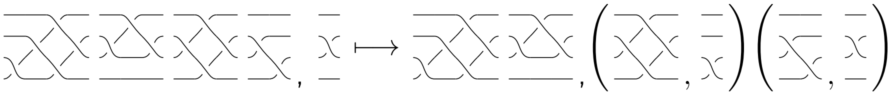
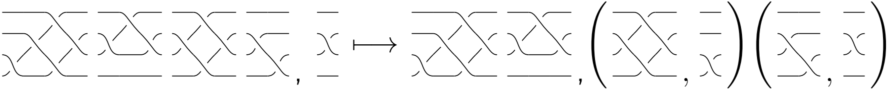
Taking the maximal penetration sequence gives an injective map

$$X_{i,k} \to \mathcal{L}^{(k-i)} \times \mathsf{PSeq}_i$$

Criterion for bounded expected pd (cont.)



Criterion for bounded expected pd (cont.)



Sketch proof (cont.)

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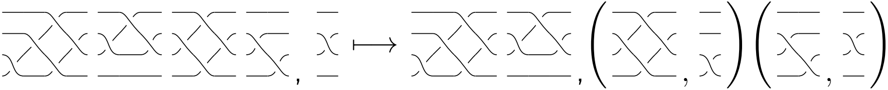
Criterion for bounded expected pd (cont.)

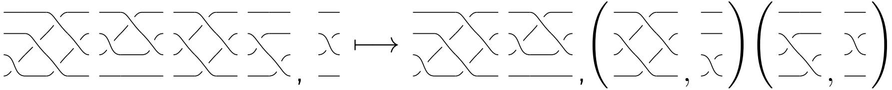


Sketch proof (cont.)

Hence

$$\mathbf{E}_{\nu_k \times \mu_{\mathcal{A}}}[\mathrm{pd}] \leqslant \sum_{i=0}^{k} i \, \frac{|\mathcal{L}^{(k-i)}| \cdot |\mathsf{PSeq}_i|}{|\mathcal{L}^{(k)}| \cdot |\mathcal{A}|}$$

Criterion for bounded expected pd (cont.)



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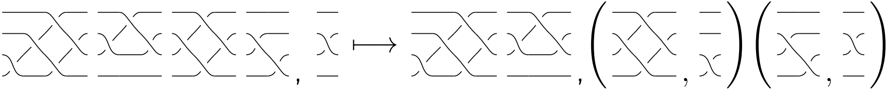
Criterion for bounded expected pd (cont.)



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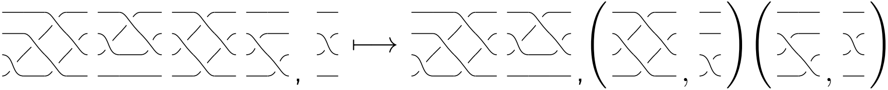
Criterion for bounded expected pd (cont.)



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$$\leqslant \sum_{i=0}^{k} i^{p+1} \left(\frac{\alpha}{\beta}\right)^i \to C < \infty \quad \text{as } k \to \infty$$

□

Criterion for unbounded expected pd

Theorem

Let ν_k be the uniform probability measure on $\mathcal{L}^{(k)}$. If $\Gamma \setminus \{1_\Gamma\}$ is strongly connected and $\alpha = \beta$ holds, then the expected value $\mathbf{E}_{\nu_k \times \mu_\mathcal{A}}[\mathrm{pd}]$ of the penetration distance with respect to $\nu_k \times \mu_\mathcal{A}$ tends to ∞.

$$\lim_{k \to \infty} \mathbf{E}_{\nu_k \times \mu_\mathcal{A}}[\mathrm{pd}] = \infty$$

- Γ is the acceptor for the language $\mathcal{L}$ of normal forms.

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and further generalised to *Garside monoids*.

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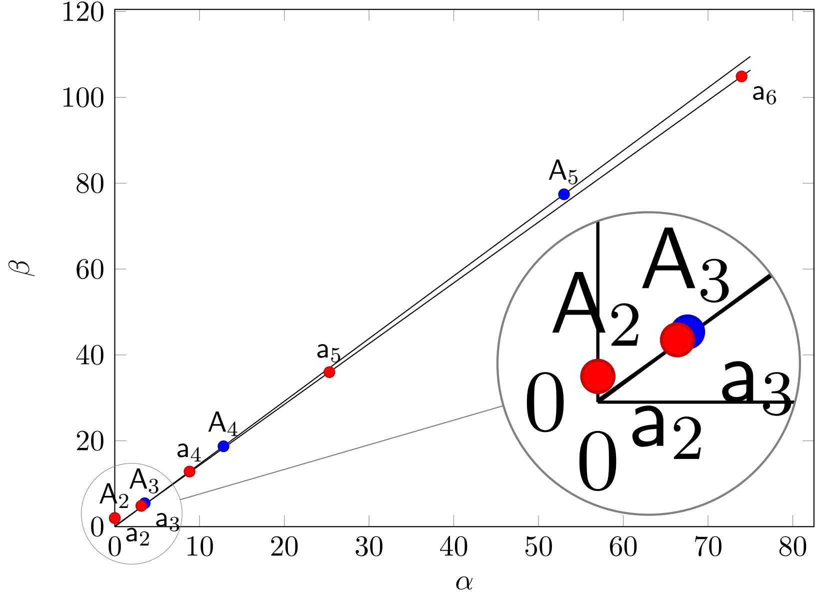
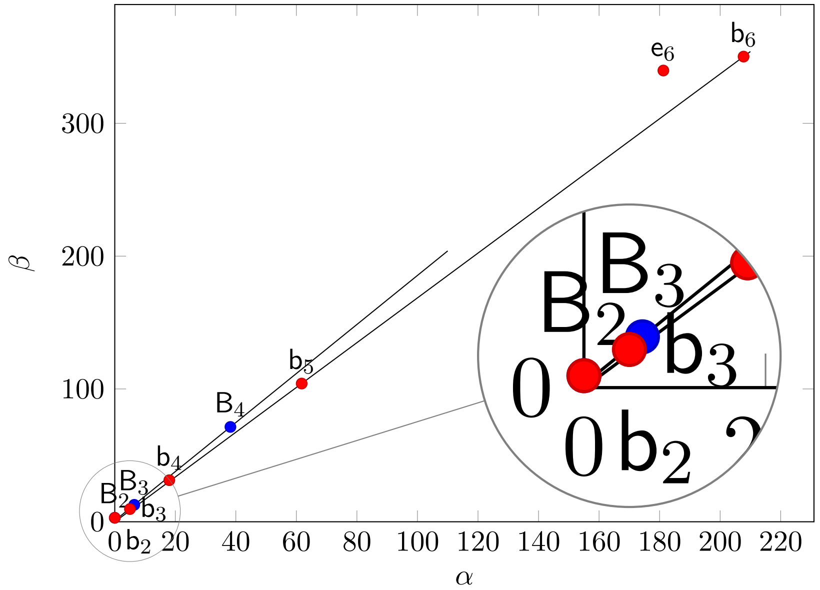
- For (slightly) larger n, there's an algorithm which can compute α and β to within a prescribed error bound.

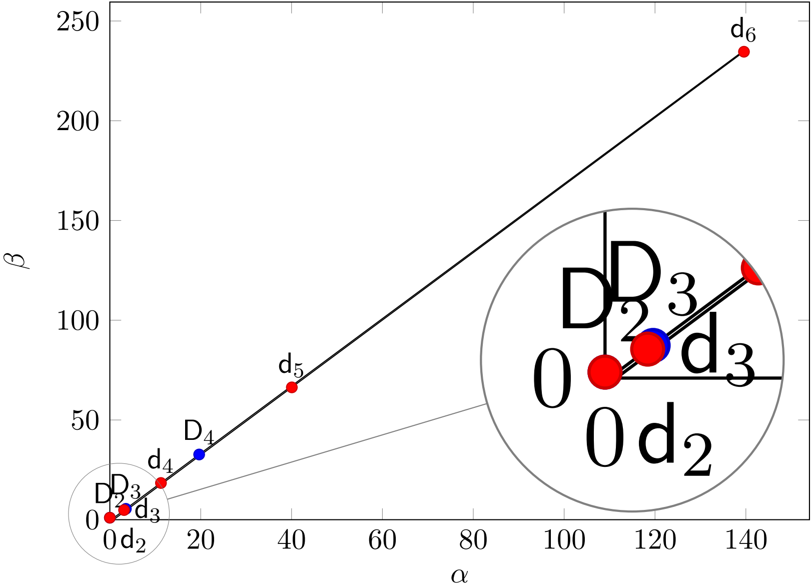
Growth rates for type A



Growth rates for types B and E



Growth rates for type D



Growth rates for types F and H